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Electromagnetic Waves in a Three-Dimensional Half Space with a Dissipative Boundary*

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We extend to dissipative problems the technique developed in [5, 12–14] for representing solutions of selfadjoint plane-boundary problems. It is clear that the method developed below is applicable to dissipative problems for an entire class of first-order symmetric-hyperbolic systems [4], but to avoid the stale format of strings of formal hypotheses we prefer to illustrate the method for a simple example of physical importance: the Cauchy problem for Maxwell's equations in a half space with the Leontovich boundary condition. This problem is solved by a (C_0) contractive semigroup for which we obtain an explicit representation in the form of superpositions of reflected plane-wave and surface modes (Theorem 4.16).

To formulate the problem, let $E = \operatorname{diag}(\varepsilon I_3, \mu I_3)$ be the diagonal 6×6 matrix with diagonal elements ε, μ which are the electromagnetic constants of a homogeneous, isotropic, loss-free medium filling the half space $R^3_{-a} = \{x \in R^3 : x_3 > -a\}$, $a \geqslant 0$. Let $A(D)$, $D_j = -i\partial_j$, $j = 1, 2, 3$, be the 6×6 matrix operator

$$A(D) = \sum_{j=1}^{3} A_j D_j = \begin{pmatrix} 0 & i \operatorname{rot} \\ -i \operatorname{rot} & 0 \end{pmatrix}, \qquad \operatorname{rot} = \begin{pmatrix} 0 & -\partial_3 & \partial_2 \\ \partial_3 & 0 & -\partial_1 \\ -\partial_2 & \partial_1 & 0 \end{pmatrix}. \tag{0.1}$$

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Maxwell's equations for the half space R^3_{-a} can now be written

$$i\partial_t f(x,t) = E^{-1}A(D)f(x,t) \equiv \mathcal{A}(D)f(x,t), \qquad x \in R^3_{-a}, \quad t>0, \tag{0.2}$$

where $\mathcal{A}(D) = E^{-1}A(D)$, and $f = \binom{f^1}{f^2}$ is a pair of column vectors: $f^1 = {}^t(f_1, f_2, f_3) \equiv E$ is the electric field and $f^2 = {}^t(f_4, f_5, f_6) \equiv H$ is the magnetic field. The problem is now to find a solution of (0.2) which is square summable on $x \in R^3_{-a}$ for each t and satisfies the initial and boundary conditions

$$f(x,0) = f_0(x), \qquad x \in R^3_{-a}, \tag{0.3}$$

$$f_1(x',-a,t) + \alpha f_5(x',-a,t) = f_2(x',-a,t) - \alpha f_4(x',-a,t) = 0 \tag{0.4}$$

$$x' = (x_1, x_2) \in R^2, \qquad t>0, \quad \alpha \in C, \quad \operatorname{Re}\alpha \geqslant 0.$$

In physics the parameter α in (0.4) is proportional to one over the square root of the complex dielectric permittivity of a conducting medium filling the half space $\{x \in R^3 : x_3 < -a\}$ [2, 8], and the intention of boundary condition (0.4) is to replace the inhomogeneous initial value problem in R^3 by the homogeneous initial boundary value problem (0.2)–(0.4) thought to approximate this problem in the half space R^3_{-a}. A heuristic discussion of the range of parameters over which this approximation might be satisfactory can be found in [1, 2]; we discuss this briefly in Section 6. We mention also that in real media α is, in general, frequency dependent with $\operatorname{Re}\alpha > 0$ and $\operatorname{Im}\alpha \neq 0$. Our results pertain only to constant α satisfying these conditions.

The content of the paper can be outlined as follows. After introducing the necessary background material in Section 1, in Section 2 we show that the differential operator $\mathcal{A}(D)$ of (0.2) with boundary condition (0.4) engenders a maximal-dissipative operator $\mathcal{A}$ in the appropriate Hilbert space. The Hille–Yosida theorem then implies that the solution of problem (0.2)–(0.4) is delivered by a (C_0) contractive semigroup $S(t) = \exp(-i\mathcal{A}t)$. In Section 3 we construct the resolvent $\mathcal{S}(\zeta) = (\mathcal{A} - \zeta I)^{-1}$ of the operator $\mathcal{A}$, and in Section 4 we compute the integral of $\mathcal{S}(\zeta)$ about the spectrum of $\mathcal{A}$ in such a way that it yields a representation of $S(t)$. Section 5 contains a description of the structure of the surface waves: they consist of second-order differential operators applied to solutions of the scalar-wave equation in R^3_{-a} which satisfy dissipative boundary conditions on $\{x_3 = -a\}$. Section 6 contains a brief discussion of the technique in application to past, present, and future problems. We remark that the modifications of the previous work [12–14] required here provide notable simplifications even in the selfadjoint case. In particular, it is now possible to compute explicitly the projections corresponding to imbedded eigenvalues (the eigenvalue $\zeta = 0$ in the present case); aside from providing a more complete Parseval identity, this eliminates the nuisance of working on the orthogonal complement of the subspaces corresponding to such eigenvalues and hence affords considerable technical simplification.

1. Background and Notation

We recall some properties of the operator $A(D)$ of (0.2) that are needed below. More details can be found in [11–14].

The transpose of a matrix is denoted by ${}^tM$ and the conjugate transpose by ${}^{\bar{t}}M$, while the adjoint of a matrix or operator is denoted by M^*. With respect to the E inner product (see (0.2)) in C^6, $(\alpha, \beta) = {}^{\bar{t}}\alpha E\beta$, the symbol $A(\eta)$, $\eta \in R^3\backslash\{0\}$,

$$A(\eta) = \begin{pmatrix} 0 & -\varepsilon^{-1}\eta\wedge \\ \mu^{-1}\eta\wedge & 0 \end{pmatrix}, \qquad \eta\wedge = \begin{pmatrix} 0 & -\eta_3 & \eta_2 \\ \eta_3 & 0 & -\eta_1 \\ -\eta_2 & \eta_1 & 0 \end{pmatrix}, \tag{1.1}$$

is selfadjoint with distinct eigenvalues $\lambda_j(\eta) = jc\,|\eta|$, $j = 0, \pm 1$, $c^2 = (\varepsilon\mu)^{-1}$, each of multiplicity two. The resolution of the identity for $A(\eta)$ is

$$\begin{aligned} I &= P_0(\omega) + P_1(\omega) + P_{-1}(\omega), \\ A(\eta) &= \lambda_1(\eta)\,P_1(\omega) + \lambda_{-1}(\eta)\,P_{-1}(\omega), \qquad \omega = \eta/|\eta|, \end{aligned} \tag{1.2}$$

where the $P_k(\omega)$ are mutually orthogonal orthoprojectors with respect to the E inner product

$${}^{\bar{t}}(EP_k) = EP_k, \qquad \delta_{jk}P_k = P_jP_k, \qquad j, k = 0, \pm 1, \tag{1.3}$$

$$\begin{aligned} P_0(\omega) &= \begin{pmatrix} t_{\omega\omega} & 0 \\ 0 & t_{\omega\omega} \end{pmatrix} = |\eta|^{-2} \begin{pmatrix} t_{\eta\eta} & 0 \\ 0 & t_{\eta\eta} \end{pmatrix}, \\ P_j(\omega) &= 2^{-1} \begin{pmatrix} -\omega\wedge\omega\wedge & -j\mu c\omega\wedge \\ j\varepsilon c\omega\wedge & -\omega\wedge\omega\wedge \end{pmatrix} \\ &= 2^{-1}\lambda_j^{-1}(\eta)\,|\eta|^{-2} \begin{pmatrix} -\lambda_j(\eta)\,\eta\wedge\eta\wedge & -\varepsilon^{-1}\,|\eta|^2\,\eta\wedge \\ \mu^{-1}\,|\eta|^2\,\eta\wedge & -\lambda_j(\eta)\,\eta\wedge\eta\wedge \end{pmatrix}, \\ \omega\wedge\omega\wedge &= (\omega\wedge)^2, \qquad \omega = \eta/|\eta| \in S^2, \qquad j = \pm 1. \end{aligned} \tag{1.4}$$

Let $\mathscr{H}_n$ denote the space of functions $f, g \in L_2(R^n, C^6)$ with the E inner product $(f, g)_{\mathscr{H}_n} = \langle f, Eg\rangle$, where here and below $\langle\cdot, \cdot\rangle$ is the usual L_2 inner product; the norm in L_2 we denote by $\|\cdot\|$. Let $\mathscr{S}'$ denote the dual of the space $\mathscr{S} = \mathscr{S}(R^n, C^6)$ of rapidly decreasing, smooth functions. The Fourier transform

$$\Phi_n f(\eta) = (2\pi)^{-n/2} \int_{R^n} \exp(-i\eta x) f(x)\, dx \equiv \hat{f}(\eta)$$

is an automorphism of $\mathscr{S}$ with inverse $\Phi_n^* f(p) = \Phi_n f(-p)$ which extends by continuity to an automorphism of $L_2(R^n, C^6)$ and by duality to an automorphism of $\mathscr{S}'$. (Below Φ with no subscript means Φ_3). For any $f \in \mathscr{H}_3$ the quantity $A(D) f \in \mathscr{S}'$, and the operator A with domain $\mathscr{D}(A) = \{f \in \mathscr{H}_3 : Af \in \mathscr{H}_3\}$ is selfadjoint with resolvent $I(\zeta) = (A - \zeta I)^{-1}$ given by

$$I(\zeta) f(x) = \int_{R^3} I(x, y; \zeta) f(y)\, dy, \qquad \operatorname{Im} \zeta \neq 0, \tag{1.5}$$

where $I(x, y; \zeta) = I(x - y; \zeta)$ is the fundamental solution,

$$[A(D) - \zeta I]\, I(x; \zeta) = \delta(x) I \tag{1.6}$$

obtained from

$$[A(\eta) - \zeta I]^{-1} = \sum_{k=-1}^{1} [\lambda_k(\eta) - \zeta]^{-1} P_k(\omega) \tag{1.7}$$

by Fourier transform in $\mathscr{S}'$

$$I(\cdot\,; \zeta) = (2\pi)^{-3/2}\, \Phi^* [A(\cdot) - \zeta I]^{-1}\, \Phi. \tag{1.8}$$

The construction of the resolvent kernels in Section 3 is based on a simple, explicit formula for $I(x, y; \zeta)$ on any hyperplane $x_3 = \text{const}$ which we now proceed to obtain. Writing $R^3 \ni \eta = (\xi, \rho)$, $\xi \in R^2$, we extend $\lambda_j(\eta) = jc\sqrt{(|\xi|^2 + \rho^2)}$ to complex $\tau = \rho + i\kappa$ by $\lambda_j(\xi, \tau) = c\sqrt{(\tau^2 + |\xi|^2)}$, $j \operatorname{Re} \lambda_j(\xi, \tau) \geqslant 0$, $j = \pm 1$. Replacing η by (ξ, τ) and $\lambda_j(\eta)$ by $\lambda_j(\xi, \tau)$ in (1.4), the $P_k(\xi, \tau)$ satisfy $A(\xi, \tau) P_k(\xi, \tau) = \lambda_k(\xi, \tau) P_k(\xi, \tau)$, $k = 0, \pm 1$. For $|\xi| \neq 0$ we define

$$\tau_\pm(\xi, \zeta) = c^{-1} \sqrt{(\zeta^2 - c^2 |\xi|^2)}, \qquad \pm \operatorname{Im} \tau_\pm \geqslant 0, \tag{1.9}$$

in the ζ plane with branch cuts $(-\infty, -c|\xi|)$, $(c|\xi|, \infty)$; observe that

$$\tau_\pm(\xi, \zeta) = -\tau_\mp(\xi, \zeta), \qquad \bar{\tau}_\pm(\xi, \bar{\zeta}) = -\tau_\pm(\xi, \zeta). \tag{1.10}$$

For $|\xi|\, \zeta \neq 0$ the matrix $[A(\xi, \tau) - \zeta I]^{-1}$ is regular in τ except for poles in the upper- (lower-) half plane at the zeros of $\det[A(\xi, \tau) - \zeta I]$, i.e., at $\tau_+ (\tau_-)$, and in a neighborhood of these poles

$$[A(\xi, \tau) - \zeta I]^{-1} = \sum_{k=-1}^{1} [\lambda_k(\xi, \tau) - \zeta]^{-1} P_k(\xi, \tau). \tag{1.11}$$

Applying Φ_2 to (1.8), we obtain in $\mathscr{S}'$

$$\Phi_2 I(\xi, x_3, y; \zeta) = (2\pi)^{-2} e^{-iy'\xi} \int_{-\infty}^{\infty} e^{i(x_3-y_3)\tau} [A(\xi, \tau) - \zeta I]^{-1} d\tau. \quad (1.12)$$

Evaluating by the residue theorem, we obtain the following expression needed below; here and henceforth we write $\tau \equiv \tau_+$ in order to simplify notation:

$$\begin{aligned} \Phi_2 I(\xi, -a, y; \zeta) &= \beta(\xi, y; \zeta)\, P(\xi, \zeta, -\tau), \qquad -a < y_3 < \infty, \\ \beta(\xi, y; \zeta) &= i(2\pi)^{-1} c^{-2}\zeta\tau^{-1} \exp[-iy'\xi + i\tau(a + y_3)]. \end{aligned} \quad (1.13)$$

Here $P(\xi, \zeta, \tau) = P^2(\xi, \zeta, \tau)$ is given explicitly by

$$P(\xi, \zeta, \tau) = 2^{-1}c^2\zeta^{-2} \begin{pmatrix} -(\xi, \tau)\wedge(\xi, \tau)\wedge & -\mu\zeta(\xi, \tau)\wedge \\ \varepsilon\zeta(\xi, \tau)\wedge & -(\xi, \tau)\wedge(\xi, \tau)\wedge \end{pmatrix}, \quad (1.14)$$

$$(\xi, \tau)\wedge = \begin{pmatrix} 0 & -\tau & \xi_2 \\ \tau & 0 & -\xi_1 \\ -\xi_2 & \xi_1 & 0 \end{pmatrix},$$

and it is a solution of

$$A(\xi, \tau)\, P(\xi, \zeta, \tau) = \zeta P(\xi, \zeta, \tau). \quad (1.15)$$

We observe that

$$^{t}[EP(\xi, \bar{\zeta}, \tau(\xi, \bar{\zeta}))] = EP(\xi, \zeta, -\tau(\xi, \zeta)). \quad (1.16)$$

2. The Maximal Dissipative Operator Λ in $\mathscr{H}$

In this section we form an operator A and its adjoint A^* which are maximal dissipative in the space $\mathscr{H}$ consisting of functions f, $g \in L_2(R^3_{-a}, C^6)$ with the E inner product (see (0.2))

$$(f, g) = \int_{R^3_{-a}} {}^{t}\bar{f}(x)\, Eg(x)\, dx.$$

The operators A, A^* are engendered by the operator $A(D)$ of (0.2) and the respective boundary conditions

$$Bf(x', -a) = 0, \qquad B = \begin{pmatrix} 1 & 0 & 0 & 0 & \alpha & 0 \\ 0 & 1 & 0 & -\alpha & 0 & 0 \end{pmatrix}, \quad (2.1)$$

$$B'f(x', -a) = 0, \qquad B' = \begin{pmatrix} 1 & 0 & 0 & 0 & -\bar{\alpha} & 0 \\ 0 & 1 & 0 & \bar{\alpha} & 0 & 0 \end{pmatrix}. \quad (2.1')$$

Here, $\alpha = \alpha_1 + i\alpha_2 \in C$, $\operatorname{Re}\alpha = \alpha_1 \geqslant 0$. Setting $\mathscr{B} = \ker B \subset C^6$, it is easily seen that $\mathscr{B}' = \ker B' = [A_3 \mathscr{B}]^\perp$, where (cf. (0.1))

$$A_3 f = {}^t(f_5, -f_4, 0, -f_2, f_1, 0), \tag{2.3}$$

and for $a \in \mathscr{B}$, $b \in \mathscr{B}'$

$$\langle a, A_3 b\rangle = {}^t\bar{a} A_3 b = 0. \tag{2.4}$$

For the properties of dissipative operators which are used below see, e.g., [7, 10]. We remark that the dissipative operator A of [7, 10] will here be $A = -i\mathcal{A}$, $A^* = i\mathcal{A}^*$, so that, e.g., the dissipative conditions used there become in our case

$$0 \geqslant \operatorname{Re}(f, Af) \;= (-i/2)[(f, \mathcal{A}f) - (\mathcal{A}f, f)] \;\;= \operatorname{Im}(f, \mathcal{A}f), \tag{2.5}$$

$$0 \geqslant \operatorname{Re}(f, A^*f) = (-i/2)[(\mathcal{A}^*f, f) - (f, \mathcal{A}^*f)] = \operatorname{Im}(\mathcal{A}^*f, f). \tag{2.5'}$$

DEFINITION 2.1. The operators $\mathring{\mathcal{A}}$, $\mathring{\mathcal{A}}'$ in $\mathscr{H}$ are $\mathcal{A}(D)$ on the respective domains

$$\mathscr{D}(\mathring{\mathcal{A}}) = \{f \in \mathscr{D}(\overline{R^3_{-a}}, C^6) \colon Bf(x', -a) = 0\}, \tag{2.6}$$

$$\mathscr{D}(\mathring{\mathcal{A}}') = \{f \in \mathscr{D}(\overline{R^3_{-a}}, C^6) \colon B'f(x', -a) = 0\}, \tag{2.6'}$$

where $\mathscr{D}(\overline{R^3_{-a}}, C^6)$ is the space of smooth functions with bounded support in $\overline{R^3_{-a}}$, the closure of R^3_{-a}.

PROPOSITION 2.2. *The operators $\mathring{\mathcal{A}}$ and $\mathring{\mathcal{A}}'$ are dissipative, i.e., for $f \in \mathscr{D}(\mathring{\mathcal{A}})$ (resp. $f \in \mathscr{D}(\mathring{\mathcal{A}}')$) $\mathring{\mathcal{A}}$ ($\mathring{\mathcal{A}}'$) satisfies* (2.5) (*resp.* (2.5′)).

Proof. For $f \in \mathscr{D}(\mathring{\mathcal{A}})$ integration by parts gives

$$(f, \mathring{\mathcal{A}}f) = -i(\alpha + \bar{\alpha}) \int_{R^2} [|f_4(x', -a)|^2 + |f_5(x', -a)|^2]\, dx' + (\mathring{\mathcal{A}}f, f),$$

so that

$$\operatorname{Im}(f, \mathring{\mathcal{A}}f) = (-i/2)[(f, \mathring{\mathcal{A}}f) - (\mathring{\mathcal{A}}f, f)] = -\alpha_1 \int_{R^2} [|f_4|^2 + |f_5|^2]\, dx' \leqslant 0,$$

since $\alpha_1 \geqslant 0$. Similarly, $\operatorname{Im}(\mathring{\mathcal{A}}'f, f) \leqslant 0$.

Since $\mathscr{D}(\mathring{\mathcal{A}})$ and $\mathscr{D}(\mathring{\mathcal{A}}')$ are dense, the dissipative operators $\mathring{\mathcal{A}}$ and $\mathring{\mathcal{A}}'$ are closable, and we define $\mathcal{A}_s \equiv \bar{\mathring{\mathcal{A}}}$, $\mathcal{A}_s' \equiv \bar{\mathring{\mathcal{A}}}'$, i.e., $\mathcal{A}_s$, $\mathcal{A}_s'$ are the graph closures of $\mathring{\mathcal{A}}$, $\mathring{\mathcal{A}}'$. The operators $\mathcal{A}_s$, $\mathcal{A}_s'$ are thus closed and dissipative. This already implies the assertion of the next lemma, but we outline the proof for completeness.

LEMMA 2.3. *The ranges* $\mathscr{R}\text{an}(A_s - iI)$ *and* $\mathscr{R}\text{an}(A_s' + iI)$ *are closed subspaces of* $\mathscr{H}$.

Proof. If $f \in \mathscr{D}(A_s)$, then

$$\begin{aligned} |f|^2 &= (i/2)\{(f, [A_s - iI]f) - ([A_s - iI]f, f\} + \operatorname{Im}(f, A_s f) \\ &\leqslant |f|\,|(A_s - iI)f|, \end{aligned} \tag{2.7}$$

since A_s is dissipative. Let $F \in \overline{\mathscr{R}\text{an}(A_s - iI)}$. Then there exists a sequence $\{f_n\} \subset \mathscr{D}(A_s)$ such that $(A_s - iI)f_n \to F$ in $\mathscr{H}$. Inequality (2.7) implies that $\{f_n\}$ is Cauchy in $\mathscr{H}$; hence, $f_n \to f \in \mathscr{H}$, and thus, also, $A_s f_n \to g \in \mathscr{H}$. Since A_s is closed, $f \in \mathscr{D}(A_s)$, and $A_s f = g$; thus, $F = (A_s - iI)f$. The assertion for A_s' is proved in exactly the same way.

Our immediate objective is to show that the subspaces of Lemma 2.3 are also dense in $\mathscr{H}$ and hence that A_s and A_s' are maximal dissipative. To this end we require some technical information which may be found in [9, 11]. We denote by $L_2(R_{-a}, H^s)$ the space of functions square-integrable on $R_{-a} = (-a, \infty)$ with values in $H^s(R^2)$, $s \in R$, the usual Sobolev space. The Schwartz space of compactly supported functions on the domain $\Omega \subset R^n$ is denoted by $\mathscr{D}(\Omega)$; $C_b(\bar{R}_{-a}, H^s)$ in the space of bounded-continuous functions from $\bar{R}_{-a}$ to H^s with norm

$$|f|_b = \sup_{t \in \bar{R}_{-a}} |f(t)|_{H^s};$$

$W(R_{-a}) = \{f : f \in L_2(R_{-a}, H^0),\ \partial_t f \in L_2(R_{-a}, H^{-1})\}$ is the Hilbert space with norm

$$|f|_W^2 = |f|^2_{L_2(R_{-a}, H^0)} + |\partial_t f|^2_{L_2(R_{-a}, H^{-1})}.$$

The operator of tangential mollification is defined by $J_\varepsilon f = j_\varepsilon^* f$, the convolution of j_ε and f, where $j_\varepsilon \in \mathscr{D}(R^2)$, $j \geqslant 0$, $j(-x') = j(x')$, $\operatorname{supp} j \subset \{x' : |x'| \leqslant 1\}$, $\int j = 1$, and $j_\varepsilon(x') = \varepsilon^{-2} j(\varepsilon^{-1} x')$, $\varepsilon > 0$. We recall that $H^1(R^3_{-a})$ is isomorphic to $\{f : f \in L_2(R_{-a}, H^1),\ \partial_t f \in L_2(R_{-a}, H^0)\}$.

LEMMA 2.4 [9, 11]. *If* $f \in W(R_{-a})$ *(resp.* $f \in H^1(R^3_{-a})$*), then* $f \in C_b(\bar{R}_{-a}, H^{-1/2})$ *(resp.* $f \in C_b(\bar{R}_{-a}, H^{1/2})$*), and the mapping sending* $f \in W(R_{-a})$ *(resp.* $f \in H^1(R^3_{-a})$*) into* $f \in C_b(\bar{R}_{-a}, H^{-1/2})$ *(resp.* $f \in C(\bar{R}_{-a}, H^{1/2})$*) is continuous.*

COROLLARY 2.5 [11]. *If* $f \in W(R_{-a})$ *and* $\psi \in H^1(R^3_{-a})$, *then*

$$-i[f(-a), x(-a)] = \int_{-a}^{\infty} [D_3 f(t), \psi(t)]\, dt - \langle f, D_3 \psi \rangle, \tag{2.8}$$

where the bracket on the left side is the $H^{-1/2}$, $H^{1/2}$ *duality bracket, while that in the integrand is the duality bracket for* H^{-1}, H^{1}. *Moreover,* $f \in W(R{-}a)$ *implies that* $J_{1/j}f = f_j \in W(R_{-a})$ *and* $f_j \to f$ *in* $W(R_{-a})$ *as* $j \to \infty$; *hence, on any hyperplane* $x_3 = c$ *also* $f_j(\cdot, c) \to f(\cdot, c)$ *in* $H^{-1/2}$.

LEMMA 2.6. *If* $f, g \in \mathscr{H}$ *and* $(f, \Lambda_s \phi) = (g, \phi)$ *for all* $\phi \in \mathscr{D}(\mathring{\Lambda})$, *then*

$$(-i/2)[(g, f) - (f, g)] \leqslant 0; \tag{2.9}$$

similarly, if $f, g \in \mathscr{H}$ *and* $(f, \Lambda_s' \phi) = (g, \phi)$ *for all* $\phi \in \mathscr{D}(\mathring{\Lambda}')$, *then*

$$(-i/2)[(f, g) - (g, f)] \leqslant 0. \tag{2.9$'$}$$

Proof. Set $f_j = J_{1/j} f$, $g_j = J_{1/j} g$, then $\Lambda f_j = J_{1/j} \Lambda f = g_j \in \mathscr{H}$ implies that $A_3 D_3 f_j = E g_j - A_1 D_1 f_j - A_2 D_2 f_j \in \mathscr{H}$; hence $A_3 f_j \in H^1(R^3_{-a}, C^6)$, and so $A_3 f_j$ and $\tilde{f}_j$, the component of f_j in the range of A_3, are in $C_b(\bar{R}_{-a}, H^{1/2})$ by Lemma 2.4. If $\phi \in \mathscr{D}(\mathring{\Lambda})$, then also $\phi_j = J_{1/j}\phi \in \mathscr{D}(\mathring{\Lambda})$, and hence by Corollary 2.5

$$\begin{aligned}(g_j, \phi) &= (g, \phi_j) = (f, \Lambda_s \phi_j) = (f_j, \Lambda_s \phi)\\ &= i[A_3 f_j(\cdot, -a), \phi(\cdot, -a)] + (g_j, \phi),\end{aligned} \tag{2.10}$$

so $A_3 f_j(\cdot, -a) \in \mathscr{B}^{\perp}$, and hence $f_j(\cdot, -a) \in \mathscr{B}'$. Therefore, the $H^{-1/2}$, $H^{1/2}$ duality bracket

$$[A_3 f_j, \tilde{f}_j] = 2 \operatorname{Re} \alpha \int_{R^2} \{|(f_j)_5 (x', -a)|^2 + |(f_j)_4 (x', -a)|^2\} \, dx'.$$

Hence, by Corollary 2.5

$$\begin{aligned}&(-i/2)[(g_j, f_j) - (f_j, g_j)]\\ &\quad = -\operatorname{Re} \alpha \int_{R^2} \{|(f_j)_5 (x', -a)|^2 + |(f_j)_4 (x', -a)|^2\} \, dx' \leqslant 0.\end{aligned} \tag{2.11}$$

The proof of the second assertion is essentially the same, where now (2.10) implies that $\tilde{f}_j \in \mathscr{B}$, because $[A_3(A_3 \mathscr{B})^{\perp}]^{\perp} = \mathscr{B}$. The $H^{-1/2}$, $H^{1/2}$ duality bracket is now

$$[A_3 f_j, \tilde{f}_j] = -2 \operatorname{Re} \alpha \int_{R^2} \{|(f_j)_5 (x', -a)|^2 + |(f_j)_4 (x', -a)|^2\} \, dx',$$

and (2.9′) now follows from Corollary 2.5.

THEOREM 2.7. *The operators* Λ_s and Λ_s' *are maximal dissipative, and* $\Lambda_s' = \Lambda_s^*$.

Proof. To prove the first assertion it suffices to show that the two subspaces of Lemma 2.3 are dense in $\mathscr{H}$ (and so equal to $\mathscr{H}$). Suppose that $f \in \mathscr{H}$ is orthogonal to $\mathscr{R}\mathrm{an}(A_s - iI)$. Then $(f, A_s\phi) = (-if, \phi)$ for all $\phi \in \mathscr{D}(\mathring{A})$, and hence by Lemma 2.6, inequality (2.9)

$$0 \geqslant (-i/2)[(-if, f) - (f, -if)] = |f|^2,$$

and so $f = 0$. In the same way, inequality (2.9′) implies that $\mathscr{R}\mathrm{an}(A_s' + iI)$ is dense. If $\phi \in \mathscr{D}(\mathring{A}')$, $\psi \in \mathscr{D}(\mathring{A})$ integration by parts and (2.4) show that $(\mathring{A}\psi, \phi) = (\psi, \mathring{A}'\phi)$, and this equality then holds for any $f \in \mathscr{D}(A_s)$. Hence, $\mathring{A}' \subset A_s^*$ and so also, $A_s' \subset A_s^*$; since A_s maximal dissipative implies that A_s^* is maximal dissipative and A_s' is maximal dissipative, it follows that $A_s' = A_s^*$.

To verify that particular functions are actually in $\mathscr{D}(A_s)$ or $\mathscr{D}(A_s^*)$ in practical calculations, it is convenient to have another characterization of the operators A_s, A_s^*. Recall now that if $f \in \mathscr{H}$ the statement that the distribution $Af = A(D)f$ is in $\mathscr{H}$ means that there exists $g \in \mathscr{H}$ such that for all $\phi \in \mathscr{D}(R^3_{-a}, C^6)$ $(g, \phi) = (f, A\phi)$.

DEFINITION 2.8. The operators A_w, A_w^* in $\mathscr{H}$ are $A(D)$ on the respective domains

$$f \in \mathscr{D}(A_w) \text{ if and only if } f,\ Af \in \mathscr{H} \text{ and there exists } g \in \mathscr{H} \text{ such that for all } \phi \in \mathscr{D}(\mathring{A}')\ (f, A\phi) = (g, \phi); \tag{2.12}$$

$$f \in \mathscr{D}(A_w') \text{ if and only if } f,\ Af \in \mathscr{H} \text{ and there exists } g \in \mathscr{H} \text{ such that for all } \phi \in \mathscr{D}(\mathring{A})\ (f, A\phi) = (g, \phi). \tag{2.12′}$$

The definition is good, because $\mathscr{D}(\mathring{A})$ and $\mathscr{D}(\mathring{A}')$ are dense in $\mathscr{H}$; also, integration by parts and (2.4) show that $\mathring{A} \subset A_w$, $\mathring{A}' \subset A_w'$.

PROPOSITION 2.9. *The operators A_w and A_w' are densely defined, closed, and dissipative.*

Proof. Let (f, g) be in the closure of the graph of A_w, and let $\{f_n\} \subset \mathscr{D}(A_w)$ be such that $f_n \to f$, $Af_n \to g$. Since $Af_n \in \mathscr{H}$, $A_3 D_3 f \in L_2(R_{-a}, H^{-1})$, and hence by Corollary 2.5 for any $\phi \in \mathscr{D}(R^3_{-a}, C^6)$

$$(f, A\phi) = \lim(f_n, A\phi) = \lim(Af_n, \phi) = (g, \phi).$$

Thus, the graph of A_w is closed, and similarly for the graph of A_w'. With $f \in \mathscr{D}(A_w)$ and $g_j = J_{1/j}Af$, we establish (2.11) in the same way as in the proof of Lemma 2.6. Passing to the limit as $j \to \infty$, we find that A_w is dissipative on the basis of Corollary 2.5.

Since $\Lambda_s \subset \Lambda_w$ and $\Lambda_s^* \subset \Lambda_w'$ we have

COROLLARY 2.10. *$\Lambda_s = \Lambda_w \equiv \Lambda$, and $\Lambda_s^* = \Lambda_w' = \Lambda_w^* \equiv \Lambda^*$.*

LEMMA 2.11. *The set $\mathscr{N} = \{{}^t(\operatorname{grad} \phi, \operatorname{grad} \psi): \psi, \phi \in \mathscr{D}(R^3_{-a})\}$ is dense in the null spaces $\mathscr{N}(\Lambda)$ and $\mathscr{N}(\Lambda^*)$ provided that $a \neq 0$, and hence $\mathscr{N}(\Lambda) = \mathscr{N}(\Lambda^*) = \bar{\mathscr{N}}$.*

Proof. If $f \in \mathscr{N}(\Lambda)$ is orthogonal to $\mathscr{N}$, then

$$\begin{aligned} \Phi_2 f(\xi, x_3) = {} & \mu(\xi)\, {}^t(\xi_1, \xi_2, i\,|\xi|, 0, 0, 0) \exp(-|\xi|\,(x_3 + a)) \\ & + \nu(\xi)\, {}^t(0, 0, 0, \xi_1, \xi_2, i\,|\xi|) \exp(-|\xi|\,(x_3 + a)), \end{aligned}$$

where μ and ν are scalar functions. The conditions that B', $B\Phi_2 f(\xi, -a) = 0$ now imply that $\mu = \nu = 0$ and hence that $f = 0$.

COROLLARY 2.12. *$f \in \mathscr{N}(\Lambda)^\perp = \mathscr{N}(\Lambda^*)^\perp \equiv \tilde{\mathscr{H}}$ if and only if*

$$\operatorname{div} f^1 = \operatorname{div} f^2 = 0. \tag{2.13}$$

Examples of functions in $\tilde{\mathscr{H}}$ are those in

$$T = \{\Lambda\phi : \phi \in \mathscr{D}(R^3_{-a}, C^6)\}. \tag{2.14}$$

There are other functions of special form in $\tilde{\mathscr{H}}$ which we now construct. We define

$$d(\xi) = {}^t(\xi_1, \xi_2, i\,|\xi|), \qquad \xi \in R^2, \tag{2.15}$$

and $\underset{\sim}{0} = {}^t(0, 0, 0)$. Let μ and ν be measurable, scalar functions on R^2 such that

$$|\cdot|^{1/2}\,\mu, |\cdot|^{1/2}\,\nu \in L_2(R^2, C); \tag{2.16}$$

let $\chi_R(|\xi|)$ denote the charateristic function of the ball $\{|\xi| < R\} \subset R^2$, and define

$$\begin{aligned} \Phi_2 Z_1^R(\xi, x_3; \mu) &= \chi_R(|\xi|)\, \mu(\xi)\, {}^t(d(\xi), \underset{\sim}{0}) \exp(-|\xi|\,(x_3 + a)), \\ \Phi_2 Z_2^R(\xi, x_3; \nu) &= \chi_R(|\xi|)\, \nu(\xi)\, {}^t(\underset{\sim}{0}, d(\xi)) \exp(-|\xi|\,(x_3 + a)). \end{aligned} \tag{2.17}$$

Then there exist the limits in $\mathscr{H}$

$$Z_1(x; \mu) = \lim_{R \to \infty} Z_1^R(x; \mu), \tag{2.18$_\mu$}$$

$$Z_2(x; \nu) = \lim_{R \to \infty} Z_2^R(x; \nu). \tag{2.18$_\nu$}$$

We observe that all the components of Z_1 and Z_2 are harmonic functions on R^3_{-a}: e.g., if $\phi \in \mathscr{D}(R^3_{-a}, C)$, then for $i = 1, \dots, 6$,

$$\langle \Delta\phi, (Z_1)_i(\cdot;\mu)\rangle = \lim_{R\to\infty} \langle \Delta\phi, (Z_1)_i^R(\cdot;\mu)\rangle = 0.$$

Thus, the components are harmonic as distributions and are hence ordinary-harmonic functions [3, p. 122]. They are thus smooth functions on R^3_{-a}. Since Z_1 and Z_2 are furthermore square summable and satisfy (2.13), they are in $\tilde{\mathscr{H}}$. We observe that Z_1 is orthogonal in $\tilde{\mathscr{H}}$ to Z_2.

PROPOSITION 2.13. *Let $\bar{T}$ be the closure in $\mathscr{H}$ of the set* (2.14), *and let* $[Z_1(\cdot;\mu)]$, $[Z_2(\cdot;\nu)]$ *denote the sets of all functions of the form* (2.18_μ), (2.18_ν), *where μ and ν are measurable functions satisfying* (2.16). *Then* $[Z_1(\cdot;\mu)]$ *and* $[Z_2(\cdot;\nu)]$ *are mutually orthogonal subspaces of $\tilde{\mathscr{H}}$, and $\tilde{\mathscr{H}}$ is the E-orthogonal direct sum*

$$\tilde{\mathscr{H}} = \bar{T} \oplus [Z_1(\cdot;\mu)] \oplus [Z_2(\cdot;\nu)]. \tag{2.19}$$

Moreover, if $\tilde{\Lambda}$, $\tilde{\Lambda}^*$ are the parts of Λ, Λ^* in $\tilde{\mathscr{H}}$, $\mathscr{D}(\tilde{\Lambda}) = \tilde{\mathscr{H}} \cap \mathscr{D}(\Lambda)$, $\mathscr{D}(\tilde{\Lambda}^*) = \tilde{\mathscr{H}} \cap \mathscr{D}(\Lambda^*)$, then

$$\begin{aligned}\{0\} &= \mathscr{D}(\tilde{\Lambda}) \cap [Z_1(\cdot;\mu)] \oplus [Z_2(\cdot;\nu)] \\ &= \mathscr{D}(\tilde{\Lambda}^*) \cap [Z_1(\cdot;\mu)] \oplus [Z_2(\cdot;\nu)].\end{aligned} \tag{2.20}$$

Proof. First, Z_1 and Z_2 are orthogonal to T: e.g., for $\phi \in \mathscr{D}(R^3_{-a}, C^6)$

$$\begin{aligned} &i(\Lambda(D)\phi, Z_1(\cdot;\mu)) \\ &\quad = i\langle \Lambda(D)\phi, Z_1(\cdot;\mu)\rangle = \lim_{R\to\infty} i\langle \Lambda(D)\phi, Z_1^R(\cdot;\mu)\rangle \\ &\quad = \lim_{R\to\infty} \int_{R^3_{-a}} {}^t[\operatorname{rot}\phi^2(x)]\, \nabla\Phi_2^* \exp(-|\cdot|\,(x_3+a))\, \chi_R(|\cdot|)\, \mu(x')\, dx \\ &\quad = 0. \end{aligned}$$

Next, if $f \in \tilde{\mathscr{H}}$ is orthogonal to T, then $\operatorname{div} f^i = 0$ and $\operatorname{rot} f^i = 0$ in $\mathscr{D}'(R^3_{-a})$, $i = 1, 2$, and it is a straightforward computation to show that f has the form $f = Z_1(\cdot;\mu) + Z_2(\cdot;\nu)$, where μ and ν satisfy (2.16). If f is in the closure of $[Z_1(\cdot;\mu)]$ or $[Z_2(\cdot;\nu)]$ it is again orthogonal to T, and so f is again of the form Z_1 or Z_2. Hence, $[Z_1(\cdot;\mu)]$ and $[Z_2(\cdot;\nu)]$ are closed subspaces. Suppose that $Z_i \in \mathscr{D}(\Lambda)$; then $\Lambda Z_i \in L_2$ and $\Lambda Z_i = 0$, i.e., $(Z_i, \Lambda\phi) = 0$ for all $\phi \in \mathscr{D}(R^3_{-a})$, $i = 1, 2$. Hence, for all $\psi \in \mathscr{D}(\mathring{\Lambda}')$ by Corollary 2.5

$$0 = -i(Z_i, \Lambda\psi) = [Z_i(\cdot, -a), A_3\psi(\cdot, -a)] = [\Phi_2 Z_i(\cdot, -a), A_3\Phi_2\psi(\cdot, -a)].$$

Writing this out for $i = 1, 2$ and using the fact that $\psi \in \mathscr{D}(\mathring{A}')$ is arbitrary, we see that $\mu = \nu = 0$ a.e., and hence $Z_i = 0$.

Remark. With the set $\bar{\mathscr{N}}$ of Lemma 2.11 the space $\mathscr{H}$ thus has the decomposition

$$\mathscr{H} = \bar{\mathscr{N}} \oplus \bar{T} \oplus [Z_1(\cdot\,;\mu)] \oplus [Z_2(\cdot\,;\nu)].$$

This is, of course, just the familiar Gillberger decomposition [14]; it is the analogue for the operator A and the manifold R^3_{-a} of the perhaps more familiar Hodge decomposition.

3. The Resolvent Kernels and the Urmodes

The maximal-dissipative operators A and A^* of the preceding section generate continuous semigroups of contraction operators $S(t) = \exp(-iAt)$ and $S^*(t) = \exp(iA^*t)$ in $\mathscr{H}$. Our ultimate goal is to represent these semigroups in terms of the generalized eigenfunctions of A and A^*. To this end we need a representation of the resolvents $\mathscr{G}(\zeta) = (A - \zeta I)^{-1}$ and $\mathscr{G}^*(\zeta) = (A^* - \bar{\zeta} I)^{-1}$ of A and A^* in terms of resolvent kernels $G(x, y; \zeta)$, $G'(x, y; \bar{\zeta})$: for $f \in \mathscr{H}$

$$\mathscr{G}(\zeta) f(x) = \int_{R^3_{-a}} G(x, y; \zeta) f(y)\, dy,$$

$$\mathscr{G}^*(\zeta) f(x) = \int_{R^3_{-a}} G'(x, y; \bar{\zeta}) f(y)\, dy.$$

The present section is devoted to obtaining a representation of these kernels.

First, we observe that the fact that $\mathscr{G}^*(\zeta)$ is the adjoint of $\mathscr{G}(\zeta)$ implies that the kernels are related by

$$EG'(x, y; \bar{\zeta}) = {}^{\mathrm{t}}[EG(y, x; \zeta)]. \tag{3.2}$$

In the selfadjoint cases $\alpha = 0$ or $\operatorname{Re}\alpha = 0$, $\operatorname{Im}\alpha \neq 0$, $G' = G$, and this becomes the usual relation $EG'(x, y; \bar{\zeta}) = EG(x, y; \bar{\zeta}) = {}^{\mathrm{t}}[EG(y, x; \zeta)]$.

We seek G and G' in the form

$$\begin{aligned} G(x, y; \zeta) &= I(x, y; \zeta) - R(x, y; \zeta) \\ G'(x, y; \bar{\zeta}) &= I(x, y; \bar{\zeta}) - R'(x, y; \bar{\zeta}), \qquad x, y \in R^3_{-a}, \end{aligned} \tag{3.3}$$

where $I(x, y; \zeta)$ of (1.5) can be interpereted as the radiation incident at $x \in R^3_{-a}$ due to a unit source at the point y, and R and R' are the fields reflected from the boundary $\{x_3=-a\}$. The functions R and R' should satisfy

$$\begin{aligned} &[\Lambda(D_x)-\zeta I]\, R(x,y;\zeta)=0,\\ &[\Lambda(D_x)-\bar{\zeta} I]\, R'(x,y;\bar{\zeta})=0, \qquad x,y\in R^3_{-a}, \end{aligned} \tag{3.4}$$

and the boundary conditions on $\{x_3=-a\}$ (see (2.1), (2.1′))

$$\begin{aligned} BR(x',-a,y;\zeta)&=BI(x',-a,y;\zeta),\\ B'R'(x',-a,y;\bar{\zeta})&=B'I(x',-a,y;\bar{\zeta}). \end{aligned} \tag{3.5}$$

Taking the Fourier transform on x', Eqs. (3.4) and (3.5) become

$$[\Lambda(\xi,D_{x_3})-\zeta I]\,\Phi_2 R(\xi,x_3,y;\zeta)=0, \tag{3.6}$$

$$[\Lambda(\xi,D_{x_3})-\bar{\zeta} I]\,\Phi_2 R'(\xi,x_3,y;\bar{\zeta})=0, \tag{3.7}$$

$$B\Phi_2 R(\xi,-a,y;\zeta)=B\Phi_2 I(\xi,-a,y;\zeta), \tag{3.8}$$

$$B'\Phi_2 R'(\xi,-a,y;\bar{\zeta})=B'\Phi_2 I(\xi,-a,y;\bar{\zeta}). \tag{3.9}$$

Equations (3.6) and (3.7) are satisfied by (see (1.15))

$$\Phi_2 R(\xi,x_3,y;\zeta)=\beta(\xi,y;\zeta)\exp(i\tau(x_3+a))\,P(\xi,\zeta,\tau)\,C(\xi,\zeta), \tag{3.10}$$

$$\begin{aligned} &\Phi_2 R'(\xi,x_3,y;\bar{\zeta})=\beta(\xi,y;\zeta)\exp(i\tau(\xi,\bar{\zeta})(x_3+a))\,P(\xi,\bar{\zeta},\tau(\xi,\bar{\zeta}))\,C'(\xi,\bar{\zeta}),\\ &\beta(\xi,y;\zeta)=i(2\pi)^{-1}c^{-2}\zeta\tau^{-1}\exp(-iy'\xi+i\tau(y_3+a)), \quad \tau=\tau(\xi,\zeta), \end{aligned}$$

where $C(\xi,\zeta)$ and $C'(\xi,\bar{\zeta})$ are chosen to satisfy (3.8), (3.9) which are now (see (1.13))

$$\begin{aligned} BP(\xi,\zeta,-\tau(\xi,\zeta))&=BP(\xi,\zeta,\tau(\xi,\zeta))\,C(\xi,\zeta),\\ B'P(\xi,\bar{\zeta},-\tau(\xi,\bar{\zeta}))&=B'P(\xi,\bar{\zeta},\tau(\xi,\bar{\zeta}))\,C'(\xi,\bar{\zeta}). \end{aligned} \tag{3.11}$$

Using (1.14), we obtain the solutions

$$\begin{aligned} &C(\xi,\zeta)=(\Delta(\xi,\zeta))^{-1}\,[\varepsilon\zeta\tau(\xi,\zeta)(\mu\varepsilon^{-1}-\alpha^2)\,Q_a+\alpha C(\xi)],\\ &C'(\xi,\zeta)=(\Delta'(\xi,\zeta))^{-1}\,[\varepsilon\bar{\zeta}\tau(\xi,\bar{\zeta})(\mu\varepsilon^{-1}-\bar{\alpha}^2)\,Q_a-\bar{\alpha}C(\xi)],\\ &Q_a=\operatorname{diag}(1,1,-1,-1,-1,1),\\ &C(\xi)=\begin{pmatrix} c(\xi) & 0\\ 0 & c(\xi)\end{pmatrix}, \qquad c(\xi)=\begin{pmatrix} \xi_2^2-\xi_1^2 & -2\xi_1\xi_2 & 0\\ -2\xi_1\xi_2 & \xi_1^2-\xi_2^2 & 0\\ 0 & 0 & |\xi|^2\end{pmatrix} \end{aligned} \tag{3.12}$$

$$\Delta(\xi, \zeta) = \Delta_E(\xi, \zeta)\, \Delta_M(\xi, \zeta), \qquad \Delta'(\xi, \bar{\zeta}) = \Delta'_E(\xi, \bar{\zeta})\, \Delta'_M(\xi, \bar{\zeta}), \tag{3.13}$$

$$\Delta_E(\xi, \zeta) = \mu\zeta + \alpha\tau, \qquad \Delta_M(\xi, \zeta) = \alpha\varepsilon\zeta + \tau, \tag{3.14}$$

$$\Delta'_E(\xi, \bar{\zeta}) = \mu\bar{\zeta} - \bar{\alpha}\tau(\xi, \bar{\zeta}) \quad = \bar{\Delta}_E(\xi, \zeta), \tag{3.15}$$

$$\Delta'_M(\xi, \bar{\zeta}) = -\bar{\alpha}\varepsilon\bar{\zeta} + \tau(\xi, \bar{\zeta}) = -\bar{\Delta}_M(\xi, \zeta), \tag{3.16}$$

where the second equalities in (3.15) and (3.16) follow from (1.10). We observe that

$$^{t}C(\xi, \zeta) = C'(\xi, \bar{\zeta}). \tag{3.17}$$

The symmetry relation (3.2) can be written in the equivalent form

$$E\Phi_2 G'(\xi, x_3, y; \zeta) = \exp(-i\xi(x' + y'))\, {}^{t}[E\Phi_2 G(\xi, y_3, x; \zeta)]. \tag{3.18}$$

Since from (1.13)

$$\exp(-i\xi(x' + y'))\, {}^{t}[E\Phi_2 I(\xi, y_3, x; \zeta] = E\Phi_2 I(\xi, x_3, y; \bar{\zeta}), \tag{3.19}$$

it follows that

$$\exp(-i\xi(x' + y'))\, {}^{t}[E\Phi_2 R(\xi, y_3, x; \zeta)] = E\Phi_2 R'(\xi, x_3, y; \bar{\zeta}), \tag{3.20}$$

and this with (3.17) implies the paramutation relations

$$\begin{aligned} C(\xi, \zeta)\, P(\xi, \zeta, -\tau(\xi, \zeta)) &= P(\xi, \zeta, \tau(\xi, \zeta))\, C(\xi, \zeta), \\ C'(\xi, \zeta)\, P(\xi, \zeta, -\tau(\xi, \zeta)) &= P(\xi, \zeta, \tau(\xi, \zeta))\, C'(\xi, \zeta), \end{aligned} \tag{3.21}$$

which can be checked directly.

The expressions (3.10) for R and R' are meaningless if $\zeta = \zeta(\xi)$ is a root of $\Delta(\xi, \zeta)$ (it follows from (3.13)–(3.16) that ζ is a root of $\Delta(\xi, \zeta)$ if and only if $\bar{\zeta}$ is a root of $\Delta'(\xi, \bar{\zeta})$). These eventual roots give the frequencies of the surface modes. To avoid considering various cases we henceforth assume that $\operatorname{Re} \alpha = \alpha_1 > 0$ and $\operatorname{Im} \alpha = \alpha_2 \neq 0$. The corresponding results for other values of α compatible with the condition $\operatorname{Re} \alpha_1 \geqslant 0$ are discussed in the final section.

LEMMA 3.1. *For* $\alpha = \alpha_1 + i\alpha_2$, $\alpha_1 > 0$, $\alpha_2 \neq 0$, *the functions* $\Delta_M(\xi, \zeta)$, $\Delta_E(\xi, \zeta)$ *of* (3.14) *have the respective simple roots*

$$\begin{aligned} &m(\xi) = -\varepsilon^{-1} p\,|\xi|, \quad p = p_1 + ip_2, \quad p_2 > 0, \quad \alpha_2 p_1 > 0, \quad p^2 = (\mu\varepsilon^{-1} - \alpha^2)^{-1}, \\ &e(\xi) = cq\,|\xi|, \qquad q = q_1 + iq_2, \quad q_2 < 0, \quad \alpha_2 q_1 > 0, \quad q^2 = -\alpha^2 p^2, \end{aligned} \tag{3.22}$$

for which the corresponding values of τ are

$$\begin{aligned} \tau_m(\xi) &\equiv \tau(\xi, m(\xi)) = -\alpha\varepsilon m(\xi) \quad = \alpha p\,|\xi|, \\ \tau_e(\xi) &\equiv \tau(\xi, e(\xi)) \;= -\alpha^{-1}\mu e(\xi) = -\alpha^{-1}\mu c q\,|\xi|. \end{aligned} \tag{3.23}$$

Proof. With p^2 of (3.22) from (3.14) $0 = \Delta_M(\xi, m(\xi))$ and $0 = \Delta_E(\xi, e(\xi))$ imply, respectively, that $m(\xi)^2 = \varepsilon^{-2}p^2\,|\xi|^2$ and $e(\xi)^2 = -c^2\alpha^2p^2\,|\xi|^2$, and the roots are extracted on the basis of the condition $\operatorname{Im}\tau \geqslant 0$:

$$0 \geqslant -\varepsilon^{-1}\operatorname{Im}\tau(\xi, m(\xi)) = \alpha_2 \operatorname{Re} m(\xi) + \alpha_1 \operatorname{Im} m(\xi),$$
$$0 \geqslant -\mu^{-1}\,|\alpha|^2 \operatorname{Im}\tau(\xi, e(\xi)) = \alpha_1 \operatorname{Im} e(\xi) - \alpha_2 \operatorname{Re} e(\xi).$$

The construction of the resolvent kernels (3.3) can thus be carried out for all points ζ not contained in R or the lines $e(\lambda) = cq\lambda$, $m(\lambda) = -\varepsilon^{-1}p\lambda$, $\lambda \in (0, \infty)$. We set

$$\begin{aligned} &\sigma = \sigma(A) = R \cup \{m\} \cup \{e\}, \\ &\{m\} = \{m(\lambda) = -\varepsilon^{-1}p\lambda, \lambda \in (0, \infty)\}, \qquad \{e\} = \{e(\lambda) = cq\lambda, \lambda \in (0, \infty)\}. \end{aligned} \tag{3.24}$$

Then for $\zeta \notin \sigma(A)$ $\mathscr{G}(\zeta)$, $\mathscr{G}^*(\zeta)$ can now be written in the form

$$\begin{aligned} \mathscr{G}(\zeta)\,f(x) &= \int_{R^3_{-a}} G(x, y; \zeta)\,f(y)\,dy = I(\zeta)\,f(x) - R(\zeta)\,f(x) \\ &= \int_{R^3_{-a}} I(x, y; \zeta)\,f(y)\,dy - \int_{R^3_{-a}} R(x, y; \zeta)\,f(y)\,dy, \\ \mathscr{G}^*(\zeta)\,f(x) &= \int_{R^3_{-a}} G'(x, y; \bar\zeta)\,f(y)\,dy = I(\bar\zeta)\,f(x) - R'(\bar\zeta)\,f(x) \\ &= \int_{R^3_{-a}} I(x, y; \bar\zeta)\,f(y)\,dy - \int_{R^3_{-a}} R'(x, y; \bar\zeta)\,f(y)\,dy, \end{aligned} \tag{3.25}$$

where $I(x, y; \zeta)$ is given by (1.8) and $R(x, y; \zeta)$ and $R'(x, y; \bar\zeta)$ are given by (3.10), (3.12)–(3.16).

THEOREM 3.2. *For $\zeta \notin \sigma(A)$ $\mathscr{G}(\zeta)$, $\mathscr{G}^*(\zeta)$ give the resolvents of A, A^*, i.e., for $f \in \mathscr{H}$ the expressions* (3.25) *give functions $\mathscr{G}(\zeta)f \in \mathscr{D}(A)$, $\mathscr{G}^*(\zeta)f \in \mathscr{D}(A^*)$ and $(A - \zeta I)\,\mathscr{G}(\zeta)f = f$, $(A^* - \zeta I)\,\mathscr{G}^*(\zeta)f = f$.*

Proof. We prove the assertion for $\mathscr{G}(\zeta)$; the proof for $\mathscr{G}^*(\zeta)$ is the same. It is first shown that $\mathscr{G}(\zeta)$ is a bounded operator on $\mathscr{H}$. For $\zeta \notin \sigma(A)$

$$|I(\zeta)f|_{\mathscr{H}} \leqslant |\operatorname{Im}\zeta|^{-1}\,|f|_{\mathscr{H}},$$

and

$$\Phi_2 R(\zeta) f(\xi, x_3) = ic^{-2}\zeta\tau^{-1} \exp(i\tau(x_3 + a)) P(\xi, \zeta, \tau) C(\xi, \zeta) h(\xi, \zeta),$$

where

$$h(\xi, \zeta) = \int_{-a}^{\infty} \exp(i\tau(a + y_3) \Phi_2 f(\xi, y_3) \, dy_3,$$

$$|h(\xi, \zeta)|^2 \leqslant (2 \operatorname{Im} \tau(\xi, \zeta))^{-1} \int_{-a}^{\infty} |\Phi_2 f(\xi, y_3)|^2 \, dy_3.$$

Now for large $|\xi|$, $P(\xi, \zeta, \tau) C(\xi, \zeta)$ is $O(|\xi|^2)$, and hence for a constant $c(\zeta)$

$$\int_{-a}^{\infty} |\Phi_2 R(\zeta) f(\xi, x_3)|^2 \, dx_3 = c(\zeta)(1 + |\xi|) |h(\xi, \zeta)|^2.$$

Therefore,

$$|\Phi_2 R(\zeta) f|_{\mathscr{H}}^2 \leqslant c(\zeta) \int_{R^2} (1 + |\xi|) |h(\xi, \zeta)|^2 \, d\xi$$

$$\leqslant c(\zeta) \int_{R^2} \int_{-a}^{\infty} |\Phi_2 f(\xi, y_3)|^2 \, dy_3 \, d\xi = c(\zeta) |f|_{\mathscr{H}}^2.$$

Hence, $|\mathscr{G}(\zeta) f|_{\mathscr{H}} \leqslant c(\zeta) |f|_{\mathscr{H}}$. Now let $\psi, \phi \in \mathscr{D}(R^3_{-a})$ so that $\Phi_2 \phi(\xi, x_3)$ is rapidly decreasing in ξ; then integration by parts yields $(A\psi, \mathscr{G}(\zeta)\phi) = (\psi, \phi + \zeta\mathscr{G}(\zeta)\phi)$, and hence $A\mathscr{G}(\zeta)\phi = \phi + \zeta\mathscr{G}(\zeta) \in \mathscr{H}$. Further, $B\mathscr{G}(\zeta) f(x', -a) = 0$ by construction, so $\mathscr{G}(\zeta)\phi \in \mathscr{D}(A)$, and $(A - \zeta I) \mathscr{G}(\zeta)\phi = \phi$. Now for any $f \in \mathscr{D}(A)$ let $\{\phi_n\} \subset \mathscr{D}(R^3_{-a}, C^6)$ be such that $\phi_n \to f$ in $\mathscr{H}$; then by the foregoing $\mathscr{G}(\zeta) \phi_n \in \mathscr{D}(A)$ and $(A - \zeta I) \mathscr{G}(\zeta) \phi_n = \phi_n$. Since $\mathscr{G}(\zeta)$ is bounded $\mathscr{G}(\zeta) \phi_n \to \mathscr{G}(\zeta) f$, and, since A is closed, $\mathscr{G}(\zeta) f \in \mathscr{D}(A)$ and $(A - \zeta I) \mathscr{G}(\zeta) f = f$.

Remark. Theorem 3.2 gives an alternate proof that A_s is maximal dissipative, i.e., that $\mathscr{R}\text{an}(A_s - iI) = \mathscr{H}$.

Finally, as an application of (3.17), (3.20), and (3.21), we derive expressions for the urmodes which are needed in the next section. Here and henceforth, $\chi_a(y_3) = \chi_{(-a,\infty)}(y_3)$ is the characteristic function of $(-a, \infty)$. We have

$$\chi_a(y_3) G(x, y; \zeta) = I(x, y; \zeta) - \chi_{(-\infty, -a)}(y_3) I(x, y; \zeta) - \chi_a(y_3) R(x, y; \zeta). \tag{3.26}$$

From (1.16), (3.10), and (3.17)

$$
\begin{aligned}
&{}^{t}[E\Phi_2 R'(\xi, y_3 . x; \bar{\zeta})] \\
&\quad = \bar{\beta}(\xi, x; \bar{\zeta}) \exp[i\tau(\xi, \zeta)(y_3 + a)]\, {}^{t}C'(\xi, \bar{\zeta})\, EP(\xi, \zeta, -\tau(\xi, \zeta)) \\
&\quad = \bar{\beta}(\xi, x; \bar{\zeta}) \exp[i\tau(\xi, \zeta)(y_3 + a)]\, EC(\xi, \zeta)\, P(\xi, \zeta, -\tau(\xi, \zeta));
\end{aligned}
$$

since (3.20) is equivalent to $ER(x, y; \zeta) = {}^{t}[ER'(y, x; \bar{\zeta})]$,

$$
\begin{aligned}
\Phi_{y'}^{*} R(x, \xi, y_3; \zeta) &= E^{-1}\, {}^{t}[E\Phi_{y'} R'(\xi, y_3, x; \bar{\zeta})] \\
&= \bar{\beta}(\xi, x; \bar{\zeta}) \exp[i\tau(\xi, \zeta)(y_3 + a)]\, C(\xi, \zeta)\, P(\xi, \zeta, -\tau(\xi, \zeta)),
\end{aligned}
$$

so that

$$
\begin{aligned}
&\Phi_y^{*} \chi_a(y_3)\, R(x, \eta; \zeta) \\
&\quad = \bar{\beta}(\xi, x; \bar{\zeta})\, e^{i\tau a} C(\xi, \zeta)\, P(\xi, \zeta, -\tau(\xi, \zeta)) \int_{-a}^{\infty} \exp[iy_3(\rho + \tau)]\, dy_3 \\
&\quad = -(2\pi)^{-3/2} c^{-2} \zeta \tau^{-1} \exp[-ia\rho + ix'\xi + i\tau(x_3 + a)] \\
&\qquad \times (\rho + \tau)^{-1} C(\xi, \zeta)\, P(\xi, \zeta, -\tau(\xi, \zeta)) \qquad (3.27)
\end{aligned}
$$

by (1.13) and (1.10). From (1.8)

$$
\begin{aligned}
&\Phi_{y'}^{*} \chi_{(-\infty, -a)}(y_3)\, I(x, \xi, y_3; \zeta) \\
&\quad = (2\pi)^{-2} \chi_{(-\infty, -a)}(y_3)\, e^{ix'\xi} \int_{-\infty}^{\infty} \exp[i(x_3 - y_3)\rho][A(\eta) - \zeta I]^{-1}\, d\rho;
\end{aligned}
$$

since $x_3 \in (-a, \infty)$, $\chi_{(-\infty, -a)}(y_3)(x_3 - y_3) > 0$, and so the integral may be evaluated by the residue theorem in the upper half plane with the result that

$$
\begin{aligned}
&\Phi_{y'}^{*} \chi_{(-\infty, -a)}(y_3)\, I(x, \xi, y_3; \zeta) \\
&\quad = i(2\pi)^{-1} c^{-2} \zeta \tau^{-1} \exp[ix'\xi + i(x_3 - y_3)\tau]\, \chi_{(-\infty, -a)}(y_3)\, P(\xi, \zeta, \tau).
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\Phi_y^{*} \chi_{(-\infty, -a)}(y_3)\, I(x, \eta; \zeta) \qquad (3.28) \\
&\quad = (2\pi)^{-3/2} c^{-2} \zeta \tau^{-1} \exp[ix'\xi + i\tau(x_3 + a) - ia\rho](\rho - \tau)^{-1} P(\xi, \zeta, \tau).
\end{aligned}
$$

Finally, from (1.8)

$$
\Phi_y^{*} I(x, \eta; \zeta) = (2\pi)^{-3/2} \exp(i\eta x)[A(\eta) - \zeta I]^{-1}. \qquad (3.29)
$$

Define

$$M(\eta;\zeta)=c^{-2}\zeta\tau^{-1}[(\rho-\tau)^{-1}P(\xi,\xi,\tau)-(\rho+\tau)^{-1}C(\xi,\zeta)P(\xi,\zeta,-\tau)] \tag{3.30}$$
$$M'(\eta;\zeta)=c^{-2}\zeta\tau^{-1}[(\rho-\tau)^{-1}P(\xi,\zeta,\tau)-(\rho+\tau)^{-1}C'(\xi,\zeta)P(\xi,\zeta,-\tau)].$$

From (3.26)–(3.30) we now have

$$\begin{aligned}\Psi(x,\eta;\zeta)&\equiv\exp(ia\rho)\,\Phi_y^*\chi_a(y_3)\,G(x,y;\zeta)(\eta)\,E^{-1}\\&=(2\pi)^{-3/2}\exp(ix'\xi)\{\exp[i\rho(x_3+a)][A(\eta)-\zeta I]^{-1}\\&\quad-\exp[i\tau(x_3+a)]\,M(\eta;\zeta)\}\,E^{-1}.\end{aligned} \tag{3.31}$$

The expression for $\Phi_y^*\chi_a G'(x,y;\zeta)(\eta)$ is obtained by replacing C by C' in (3.31)

$$\begin{aligned}\Psi'(x,\eta;\zeta)&\equiv\exp(ia\rho)\,\Phi_y^*\chi_a(y_3)\,\mathscr{G}'(x,y;\zeta)(\eta)\,E^{-1}\\&=(2\pi)^{-3/2}\exp(ix'\xi)\{\exp[i\rho(x_3+a)][A(\eta)-\zeta I]^{-1}\\&\quad-\exp[i\tau(x_3+a)]\,M'(\eta;\zeta)\}\,E^{-1}.\end{aligned} \tag{3.32}$$

The functions Ψ, Ψ' are the urmodes of A, A^*, respectively. Their limits as ζ approaches points of σ give the plane-wave and surface modes for A and A^*.

4. The Representation in Terms of Generalized Eigenfunctions

In this section representations of the contraction semigroups $S(t)=\exp(-itA)$, $S^*(t)=\exp(itA^*)$, $t\geqslant 0$, are obtained in terms of the generalized eigenfunctions of A, A^*. The central issue here is to establish the Parseval identity for A. A^*. Now for a selfadjoint operator S with resolvent $R_S(\zeta)$ the basis for an eventual Parseval identity for S is the well-known formula

$$I=(2\pi i)^{-1}\lim_{\kappa\downarrow 0}\int_R[R_S(\lambda+i\kappa)-R_s(\lambda-i\kappa)]\,d\lambda. \tag{4.1}$$

In order to interpret this formula in a manner that generalizes to nonselfadjoint operators, we first observe that if S is a bounded operator, (4.1) states that the integral around the spectrum of S is the identity; this is simply Cauchy's theorem in the Banach algebra of bounded operators. For unbounded operators S we can also interpret (4.1) as an integral around the spectrum in the sense that

$$I=\lim_{N\uparrow\infty}(2\pi i)^{-1}\lim_{\kappa\downarrow 0}\int_{C_N(\kappa)}R_S(\zeta)\,d\zeta, \tag{4.2}$$

where $C_N = C_N^+ \cup C_N^-$, $C_N^+ = \{\zeta = \lambda + i\kappa\colon -N < \lambda < N\}$, and $C_N^- = \{\zeta = \lambda - i\kappa\colon N > \lambda \to -N\}$; here, C_N^- is the oriented line segment from $N - i\kappa$ to $-N - i\kappa$. In the form (4.2) formula (4.1) generalizes to nonselfadjoint operators.

For fixed $\kappa > 0$ we define an oriented curve $C(\kappa)$ about the set σ of (3.24) consisting of directed line segments taken in the order indicated (see Lemma 3.1):

$$\begin{aligned}
C(\kappa) &= [i_+, i_-^r, e_+, e_-, m_+, m_-, i_-^l],\\
i_+(\kappa) &= \{\zeta = \lambda + i\kappa\colon -\infty < \lambda < \infty\},\\
i_-^r(\kappa) &= \{\zeta = \lambda - i\kappa\colon \infty > \lambda \to -\kappa q_2^{-1}(q_1 + |q|^2)\},\\
e_+(\kappa) &= \{\zeta = cq\lambda + iq\kappa\colon -(cq_2)^{-1}\,(q_1 + \kappa) \leqslant \lambda < \infty\},\\
e_-(\kappa) &= \{\zeta = cq\lambda - iq\kappa\colon \infty > \lambda \to c^{-1}\kappa(\operatorname{Im}\bar{q})^{-1}\,(|p|^2 - \operatorname{Re} p\bar{q}) > 0\},\\
m_+(\kappa) &= \{\zeta = -\varepsilon^{-1}p\lambda - i\kappa p\colon 0 < \varepsilon\kappa(\operatorname{Im} p\bar{q})^{-1}\,(|q|^2 - \operatorname{Re} p\bar{q}) \leqslant \lambda < \infty\},\\
m_-(\kappa) &= \{\zeta = -\varepsilon^{-1}p\lambda + i\kappa p\colon \infty > \lambda \to \kappa\varepsilon p_2^{-1}(p_1 + 1)\},\\
i_-^l(\kappa) &= \{\zeta = \lambda - i\kappa\colon -\kappa p_2^{-1}(p_1 + |p|^2) > \lambda \to -\infty\}.
\end{aligned} \tag{4.3}$$

To be specific, we have assumed here that $\alpha_2 = \operatorname{Im} \alpha > 0$; if $\alpha_2 < 0$ the roles of e and m are interchanged.

Let $\mathscr{H} \ni f = \chi_a F$, $F \in \mathscr{H}_3$, be the restriction to R^3_{-a} of a function $F \in \mathscr{H}_3$. From (3.27)

$$\mathscr{G}(\zeta) f(x) = \int_{R^3_{-a}} G(x, y; \zeta) f(y)\, dy \equiv \mathscr{I}(\zeta) f(x) - \mathscr{R}(\zeta) f(x), \tag{4.4}$$

where from (3.27)–(3.29)

$$\begin{aligned}
\mathscr{I}(\zeta) f(x) &= \int_{R^3_{-a}} I(x, y; \zeta)\, \chi_a F(y)\, dy = \int_{R^3} \Phi_y^* I(x, y; \zeta)(\eta)\, \Phi \chi_a F(\eta)\, d\eta\\
&= (2\pi)^{-3/2} \int_{R^3} e^{ix\eta} [A(\eta) - \zeta I]^{-1}\, \Phi \chi_a F(\eta)\, d\eta\\
&= \int_{R^3} \Phi_y^* \chi_a(y_3)\, I(x, y; \zeta)(\eta)\, \hat{F}(\eta)\, d\eta\\
&= (2\pi)^{3/2} \int_{R^3} \{e^{ix\eta} [A(\eta) - \zeta I]^{-1} - c^{-2}\zeta\tau^{-1}\\
&\qquad \times \exp[ix'\xi + i\tau(x_3 + a) - ia\rho](\rho - \tau)^{-1}\, P(\xi, \zeta, \tau)\}\, \hat{F}(\eta)\, d\eta,
\end{aligned} \tag{4.5}$$

$$\mathscr{R}(\zeta)f(x) = -(2\pi)^{-3/2}\int \{\exp[ix'\xi + i\tau(x_3+a) - ia\rho]\, c^{-2}\zeta\tau^{-1}(\rho+\tau)^{-1}C(\xi,\zeta)$$

$$\times P(\xi,\zeta,-\tau)\}\,\hat{F}(\eta)\,d\eta, \tag{4.6}$$

$$\hat{F}(\eta) \equiv \Phi F(\eta).$$

We observe that $\mathscr{I}(\zeta)f(x) = I(\zeta)\chi_a F(x)$, where $I(\zeta)$ is the free-space resolvent (1.5); $\mathscr{I}(\zeta)f(x)$ is thus actually defined for all $x \in R^3$, although expression (4.4) holds only for $x \in R^3_{-a}$.

Now let $N > N_0$, where N_0 is a fixed-positive number to be specified below, and let $C_N(\kappa)$ be the portion of the oriented curve $C(\kappa)$ of (4.3) inside a disk of radius N with center at the origin. We define

$$\begin{aligned}\mathscr{G}(N,\kappa)f(x) &= \int_{C_N(\kappa)} \mathscr{G}(\zeta)f(x)\,d\zeta \\ &= \int_{C_N(\kappa)} \mathscr{I}(\zeta)f(x)\,d\zeta - \int_{C_N(\kappa)} \mathscr{R}(\zeta)f(x)\,d\zeta \\ &\equiv \mathscr{I}(N,\kappa)f(x) - \mathscr{R}(N,\kappa)f(x).\end{aligned} \tag{4.7}$$

The Parseval identity for A essentially results from the fact that the left side of (4.7) tends weakly to f in $\mathscr{H}$ uniformly with respect to $\kappa \in (0, \kappa_0]$ as $N \to \infty$.

LEMMA 4.1. *Let $h \in \mathscr{H}_3$ and define*

$$I(N,\kappa)h = \int_{C(N,\kappa)} I(\zeta)h\,d\zeta,$$

where $I(\zeta)$ is the free-space resolvent (1.5). *Then in the topology of $\mathscr{H}_3$*

$$I(N)h \equiv \lim_{\kappa\downarrow 0} I(N,\kappa)h = 2i\pi\Phi^* \sum_{j=-1}^{1} \chi_N(\lambda_j(\,\cdot\,))\,P_j\hat{h}, \tag{4.8}$$

where $\chi_N(\cdot)$ is the characteristic function of the interval $(-N, N)$.

Proof. Since $I(\zeta)h$ is holomorphic in $C\backslash R$, in $\mathscr{H}_3$

$$I(N,\kappa)h = I'(N,\kappa)h + o_\kappa(1), \tag{4.9}$$

where

$$I'(N,\kappa)h = \int_{C'(N,\kappa)} I(\zeta)h\,d\zeta,$$

$$C'(N,\kappa) = [i'_+(N,\kappa), i'_-(N,\kappa)]$$

$$i'_{+}(N,\kappa)=\{\zeta=\lambda+i\kappa\colon -\sqrt{(N^2-\kappa^2)}<\lambda<\sqrt{(N^2-\kappa^2)}\},$$
$$i'_{-}(N,\kappa)=\{\zeta=\lambda-i\kappa\colon \sqrt{(N^2-\kappa^2)}>\lambda\to-\sqrt{(N^2-\kappa^2)}\}.$$

Further, if

$$M(N,\kappa;\eta)=\sum_{j=-1}^{1}P_j(\eta)\{\tan^{-1}\kappa^{-1}[\sqrt{(N^2-\kappa^2)}-\lambda_j(\eta)]$$
$$-\tan^{-1}\kappa^{-1}[-\sqrt{(N^2-\kappa^2)}-\lambda_j(\eta)]\},$$

then

$$\begin{aligned}&|M(N,\kappa;\eta)\,\hat{h}(\eta)|^2\leqslant 4\pi^2\,|\hat{h}(\eta)|^2,\\ &M(N,\kappa;\eta)\,\hat{h}(\eta)\to\pi\sum_{j=-1}^{1}P_j(\eta)\,\chi_N(\lambda_j(\eta))\,\hat{h}(\eta)\quad\text{pointwise as }\kappa\downarrow 0.\end{aligned}\tag{4.10}$$

Now by (1.8)

$$\begin{aligned}I'(N,\kappa)\,h(x)&=\int_{C'(N,\kappa)}\int_{R^3}I(x,y;\zeta)\,h(y)\,dy\,d\zeta\\
&=\int_{C'(N,\kappa)}\int_{R^3}\Phi_y^*I(x,y;\zeta)(\eta)\,\hat{h}(\eta)\,d\eta\,d\zeta\\
&=(2\pi)^{-3/2}\int_{C'(N,\kappa)}\int_{R^3}e^{ix\eta}[A(\eta)-\zeta I]^{-1}\hat{h}(\eta)\,d\eta\,d\zeta\\
&=(2\pi)^{-3/2}\int_{R^3}e^{ix\eta}\,d\eta\int_{-\sqrt{(N^2-\kappa^2)}}^{\sqrt{(N^2-\kappa^2)}}d\lambda\\
&\quad\times\left\{\sum_{j=-1}^{1}P_j(\eta)\,\hat{h}(\eta)\left[\frac{1}{\lambda_j(\eta)-(\lambda+i\kappa)}-\frac{1}{\lambda_j(\eta)-(\lambda-i\kappa)}\right]\right\}\\
&=(2\pi)^{-3/2}\int_{R^3}e^{ix\eta}\,d\eta\sum_{j=-1}^{1}P_j(\eta)\,\hat{h}(\eta)\int_{-\sqrt{(N^2-\kappa^2)}}^{\sqrt{(N^2-\kappa^2)}}d\lambda\\
&\quad\times\frac{2i\kappa}{[\lambda_j(\eta)-\lambda]^2+\kappa^2}\\
&=2i(2\pi)^{-3/2}\int_{R^3}e^{ix\eta}M(N,\kappa;\eta)\,\hat{h}(\eta)\,d\eta.\end{aligned}$$

Hence,

$$\begin{aligned}&\|\Phi I(N,\kappa)h-2i\pi\sum_{j=-1}^{1}P_j\chi_N(\lambda_j)\hat{h}\|^2_{\mathscr{H}_3}\\
&\quad=\int_{R^3}|2iM(N,\chi;\eta)\,\hat{h}(\eta)-2i\pi\sum_{j=-1}^{1}\chi_N(\lambda_j)\,P_j(\eta)\,\hat{h}(\eta)|^2\,d\eta+o_\kappa(1)\\
&\quad=o_\kappa(1)\end{aligned}$$

by (4.9), (4.10).

COROLLARY 4.2. *Let $g \in \mathscr{H}$, and let $\mathscr{H} \ni f = \chi_a F$ be the restriction of $F \in \mathscr{H}_3$ to R^3_{-a}. Then for $\mathscr{I}(N, \kappa) f$ of* (4.7)

$$\begin{aligned} (g, \mathscr{I}(N) f) &\equiv \lim_{\kappa \downarrow 0} (g, \mathscr{I}(N, \kappa) f) \\ &= 2i\pi \sum_{j=-1}^{1} \langle \Phi \chi_{-a}\, g, E\chi_N(\lambda_j(\cdot))\, P_j \Phi \chi_a F \rangle, \end{aligned} \tag{4.11}$$

and, hence,

$$\lim_{N \uparrow \infty} (g, \mathscr{I}(N) f) = 2i\pi (g, f). \tag{4.12}$$

Proof. For $g \in \mathscr{H}$

$$(g, \mathscr{I}(N, \kappa) f) = \langle \chi_a\, g, EI(N, \kappa)\, \chi_a F \rangle = \langle \Phi \chi_a\, g, E\Phi I(N, \kappa)\, \chi_a F \rangle.$$

Hence,

$$(g, \mathscr{I}(N) f) = \lim_{\kappa \downarrow 0} (g, \mathscr{I}(N, \kappa) f) = \langle \Phi \chi_a\, g, E\Phi I(N)\, \chi_a F \rangle.$$

Expression (4.11) now follows from (4.8), and, passing to the $\lim_{N \uparrow \infty}$, by (1.2) we obtain

$$\begin{aligned} \lim_{N \uparrow \infty} (g, \mathscr{I}(N) f) &= 2i\pi \langle \Phi \chi_a\, g, E\Phi \chi_a F \rangle \\ &= 2i\pi \langle \chi_a\, g, E\chi_a F \rangle = 2i\pi (g, f). \end{aligned}$$

LEMMA 4.3. *Let $\mathscr{H} \ni f = \chi_a F$, where $F \in \mathscr{H}_3$ is any function such that* supp $\hat{F}$ *is compact. Then for any $g \in \mathscr{H}$ and large N*

$$|(g, \mathscr{R}(N) f)| \equiv \overline{\lim_{\kappa \downarrow 0}}\, |(g, \mathscr{R}(N, \kappa) f)| = o_N(1), \tag{4.13}$$

where $o_N(1) = o_N(1; f, g)$ depends on f and g.

Proof. Let $b = \max\{1, c, c\,|q|, \varepsilon^{-1}\,|p|\}$, and fix $N > N_0$, where N_0 is such that $\operatorname{supp} \hat{F} \subset \{b\,|\eta| < N_0/4\}$ (see (4.3)). Then from (4.6) $\mathscr{R}(\zeta) f(x)$ is regular for $\zeta \in A_-(N, \kappa) = \{\zeta \in C\colon \operatorname{Im} \zeta \leqslant -\kappa,\ |\zeta| = N\}$, and, of course, it is also regular for $\zeta \in A_+(N, \kappa) = \{\zeta \in C\colon \operatorname{Im} \zeta \geqslant \kappa,\ |\zeta| = N\}$. Hence, for any $g \in \mathscr{H}$ by Cauchy's theorem

$$(g, \mathscr{R}(N, \kappa) f) = (g, \mathscr{R}'(N, \kappa) f) + o_\kappa(1), \tag{4.14}$$

where

$$\begin{aligned} \mathscr{R}'(N, \kappa) f &= \int_{A_-(N, \kappa)} \mathscr{R}(\zeta) f\, d\zeta + \int_{A_+(N, \kappa)} \mathscr{R}(\zeta) f\, d\zeta \\ &\equiv \mathscr{R}'_-(N, \kappa) f + \mathscr{R}'_+(N, \kappa) f. \end{aligned} \tag{4.15}$$

We prepare to estimate $\mathscr{R}'(N,\kappa)f$. With $\theta_0=\tan^{-1}[\kappa/\sqrt{(N^2-\kappa^2)}]$

$$A_+(N,\kappa)=\{Ne^{i\theta}: \theta_0\leqslant\theta\leqslant\pi-\theta_0\},$$
$$A_-(N,\kappa)=\{Ne^{i\theta}: \theta_0+\pi\leqslant\theta\leqslant 2\pi-\theta_0\}.$$

For $\zeta\in A_\pm(N,\kappa)$, $\eta=(\xi,\rho)\in\operatorname{supp}\hat F$

$$\begin{aligned}&4N>c\,|\tau|=|\zeta-c\,|\xi||^{1/2}\,|\zeta-c\,|\xi||^{1/2}>N/2,\\&|\rho+\tau|\geqslant N/4c.\end{aligned}\tag{4.16}$$

Now $\operatorname{Im}\tau=|\tau|\sin(\phi_1+\phi_2)/2$, where ϕ_1, ϕ_2 are the respective angles the vectors $\zeta-c\,|\xi|$, $\zeta+c\,|\xi|$ make with the real axis, $0\leqslant\phi_1<2\pi$, $-\pi\leqslant\phi_2<\pi$. From elementary geometric considerations we find that for $\zeta\in A_\pm(N,\kappa)$

$$c\operatorname{Im}\tau\geqslant N[\theta\pi^{-1}\chi_{(0,\pi/2)}(\theta)+(1-\theta\pi^{-1})\,\chi_{(\pi/2,\pi)}(\theta)],\qquad \zeta\in A_+(N,\kappa),$$
$$c\operatorname{Im}\tau\geqslant N[(2-\theta\pi^{-1})\cdot\chi_{(3\pi/2,2\pi)}(\theta)+(\theta\pi^{-1}-1)\,\chi_{(\pi,3\pi/2)}(\theta)],\quad \zeta\in A_-(N,\kappa).$$

It now immediately follows that

$$\begin{aligned}&\int_{\theta_0}^{\pi-\theta_0}\exp[-(x+a)\operatorname{Im}\tau]\,d\theta\leqslant 2\pi c/N(x+a),\qquad \zeta\in A_+(N,\kappa),\\&\int_{\pi-\theta_0}^{2\pi-\theta_0}\exp[-(x+a)\operatorname{Im}\tau]\,d\theta\leqslant 2\pi c/N(x+a),\qquad \zeta\in A_-(N,\kappa).\end{aligned}\tag{4.17}$$

Further, for all $\zeta\in A_\pm(N,\kappa)$, $\eta\in\operatorname{supp}\hat F$ for a constant $d=d(f)>0$

$$|[C(\xi,\zeta)\,P(\xi,\zeta,-\tau)]_{ij}|\leqslant d,\qquad i,j=1,\dots,6,\tag{4.18}$$

uniformly with respect to $\kappa\in(0,\kappa_0]$, and, finally, from (4.16)

$$\left|\int_{\operatorname{sup}\hat F}\exp(-ia\rho)\,\hat F(\xi,\rho)(\rho+\tau)^{-1}\,d\rho\right|^2\leqslant 2N^{-1}c\int|\hat F(\xi,\rho)|^2\,d\rho.\tag{4.19}$$

We are now ready to estimate $\mathscr{R}'_\pm(N,\kappa)$. From (4.6), (4.15) with an obviously legitimate interchange of the order of integration

$$\begin{aligned}\mathscr{R}'_\pm(N,\kappa)\,f(x)&=\int_{A_\pm(N,\kappa)}d\zeta\int_{R^3}\Phi_y^*\chi_a(y_3)\,R(x,y;\zeta)(\eta)\,\hat F(\eta)\,d\eta\\&=(2\pi)^{-3/2}\,c^{-2}\int_{R^3}\exp(ix'\xi-ia\rho)\left\{\int_{A_\pm(N,\kappa)}\exp[i\tau(x_3+a)]\right.\\&\qquad\left.\times\,\zeta\tau^{-1}(\rho+\tau)^{-1}\,C(\xi,\zeta)\,P(\xi,\zeta,-\tau)\,d\zeta\right\}\hat F(\eta)\,d\eta,\end{aligned}\tag{4.20}$$

and, hence,

$$\Phi_2 \mathscr{R}'_\pm(N,\kappa)\, f(\xi, x_3)$$

$$= (2\pi)^{-1/2}\, c^{-2} \int_{A_\pm(N,\kappa)} d\zeta \exp[i\tau(x_3 + a)]\, \zeta\tau^{-1} C(\xi,\zeta)\, P(\xi,\zeta,-\tau)$$

$$\times \int \exp(-ia\rho)(\rho+\tau)^{-1}\, \hat{F}(\xi,\rho)\, d\rho. \tag{4.21}$$

From (4.16), (4.18), (4.19), and (4.21) we now have

$$|\Phi_2 \mathscr{R}'_\pm(N,\kappa)\, f(\xi, x_3)|^2$$

$$\leqslant \pi^{-1} 2^2 d^2 N c^{-1} \left[\int_\pi \exp[-(x_3+a)\operatorname{Im}\tau]\, d\theta\right]^2 \int |\hat{F}(\xi,\rho)|^2\, d\rho$$

$$\leqslant \pi^{-1} 2^2 d^2 N c^{-1} \Big\{ \chi_{(-a,-a+1/N)}(x_3)\, \pi^2$$

$$+ \chi_{(-a+1/N,\infty)}(x_3) \left[\int_\pi \exp[-(x_3+a)\operatorname{Im}\tau|\, d\theta\right]^2 \Big\} \int |\hat{F}(\xi,\rho)|^2\, d\rho$$

$$\leqslant \pi^{-1} 2^2 d^2 c^{-1} \{ N\pi^2 \chi_{(-a,-a+1/N)}(x_3)$$

$$+ \chi_{(-a+1/N,\infty)}(x_3)\, 4\pi^2 c^2 / N(x_3+a)^2 \} \int |\hat{F}(\xi,\rho)|^2\, d\rho$$

by (4.17). Here $\int_\pi$ is $\int_0^\pi$ for $\mathscr{R}'_+$ and $\int_\pi^{2\pi}$ for $\mathscr{R}'_-$. Thus,

$$\int_{-a}^{\infty} |\Phi_2 \mathscr{R}'_\pm(N,\kappa) f(\xi, x_3)|^2\, dx_3 \leqslant 20\pi\, d^2 c \int |F(\xi,\rho)|^2\, d\rho,$$

and so

$$|\mathscr{R}'_\pm(N,\kappa) f|^2_{\mathscr{H}} \leqslant 20\pi\, d^2 c\, \|F\|^2, \tag{4.22}$$

i.e., $\mathscr{R}'_\pm(N,\kappa) f$ is uniformly bounded in $N > N_0$, $\kappa \in (0, \kappa_0]$. Next, from (4.16), (4.17), and (4.20)

$$\begin{aligned} |\mathscr{R}'_\pm(N,\kappa)\, f(x)| &\leqslant 8d(2\pi)^{-3/2} \int |\hat{F}(\eta)|\, d\eta \int_\pi \exp[-(x_3+a)\operatorname{Im}\tau]\, d\theta \\ &< 4c\, dN^{-1}(x_3+a)^{-1}\, |\hat{F}|_{L_1}. \end{aligned} \tag{4.23}$$

Now let $\phi \in \mathscr{D}(R^3_{-a}, C^6)$ (so that the distance from $\sup \phi$ to $\{x_3 = -a\}$ is positive). Then by (4.23)

$$|(\mathscr{R}'_\pm(N, \kappa)f, \phi)| < 4c\, dN^{-1}\, |\hat{F}|_{L_1} \int_{R^3_{-a}} (x_3 + a)^{-1}\, |\phi(x)|\, dx, \qquad (4.24)$$

so that (4.13) holds with $g = \phi$ by (4.14). For any $g \in \mathscr{H}$ let $\{\phi_m\} \subset \mathscr{D}(R^3_{-a}, C^6)$ be such that $\phi_m \to g$ in $\mathscr{H}$. Then by (4.22)

$$|(\mathscr{R}'_\pm(N, \kappa)f, g) \leqslant |(\mathscr{R}'_\pm(N, \kappa)f, \phi_m)| + (20\,\pi c)^{1/2} d\, \|F\|\, |g - \phi_m|.$$

Given $\delta > 0$ we choose and fix $m = m(\delta)$ such that the second term is less than $\delta/2$; using (4.24), we then choose $N > N_0 = N_0(m(\delta)) \equiv N_0(\delta)$ such that the first term is less than $\delta/2$. The left side is then less than δ for $N > N_0(\delta)$ uniformly with respect to $\kappa \in (0, \kappa_0]$. This completes the proof of Lemma 4.3.

Remark. Relation (4.12) is actually only needed below for $g \in \mathscr{D}(R^3_{-a}, C^6)$; the same applies to relation (4.25) in the next theorem.

Combining Corollary 4.2 and Lemma 4.3, we now have the following result:

THEOREM 4.4. *Let $C(N, \kappa)$ be the portion of $C(\kappa)$ of* (4.3) *inside a disk of radius N with center at the origin. Let $\mathscr{H} \ni f = \chi_a F$ be the restriction to R^3_{-a} of $F \in \mathscr{H}_3$ such that* $\operatorname{supp} \hat{F}$ *is compact. Define $\mathscr{G}(N, \kappa)f$ by* (4.7). *Then for any $g \in \mathscr{H}$*

$$\lim_{N \uparrow \infty} \overline{\lim_{\kappa \downarrow 0}}\, |(g, \mathscr{G}(N, \kappa)f) - 2i\pi(g, f)| = 0. \qquad (4.25)$$

Remark. It is seen below that for a more restricted class of f the $\lim_{\kappa \downarrow 0}(g, \mathscr{G}(N, \kappa)f)$ actually exists, so that the limit superior in (4.25) can be replaced by the ordinary limit.

The relation (4.25) is half the Parseval identity for Λ, Λ^* and this particular class of f. To obtain the other half we compute $\mathscr{G}(N, \kappa)f$ in a different way.

LEMMA 4.5. *Let $\mathscr{H} \ni f = \chi_a F$ be the restriction to R^3_{-a} of a function $F \in \mathscr{H}_3$ such that $\hat{F} \in \mathscr{D}(R^3, C^6)$ and $\{|\xi| = 0\} \cap \operatorname{supp} \hat{F} = \varnothing$. Let $0 < d =$* $\operatorname{dist}(\operatorname{supp} \hat{F},\ \{|\xi| = 0\})$ *and $4\delta = d \min\{\varepsilon^{-1}\,|p|, c\,|q|, ca_1\,|p|, c\}$. Then $\mathscr{G}(\zeta)f$ is regular in the punctured disk $D = \{|\zeta| < 2\delta\} \setminus \{0\}$. If in* (4.3) *$\kappa$ is chosen so small that the points of intersection of the line segments forming the portion of $C(N, \kappa)$ in the lower-half plane lie in D, then for any $g \in \mathscr{H}$ with $e_\pm(\lambda) =$*

$cq\lambda \pm iq\kappa$, $m_\pm(\lambda) = -\varepsilon^{-1}p\lambda \mp i\kappa p$, and $N > N_0$, where N_0 is as in the proof of Lemma 4.3,

$$\begin{aligned}(g, \mathscr{S}(N,\kappa)f) = {} & 2i\kappa \int_\delta^N (\mathscr{S}^*(\lambda \pm i\kappa)g, \mathscr{S}(\lambda \mp i\kappa)f)\, d\lambda \\ & + 2i\kappa \int_{-N}^{-\delta} (\mathscr{S}^*(\lambda \pm i\kappa)g, \mathscr{S}(\lambda \mp i\kappa)f)\, d\lambda \\ & - 2\pi i \operatorname*{Res}_{\zeta=0}(g, \mathscr{S}(\zeta)f) \\ & + 2i\kappa c q^2 \int_\delta^N (\mathscr{S}^*(e_\pm(\lambda))g, \mathscr{S}(e_\mp(\lambda))f)\, d\lambda \\ & + 2i\kappa \varepsilon^{-1} p^2 \int_\delta^N (\mathscr{S}^*(m_\pm(\lambda))g, \mathscr{S}(m_\mp(\lambda))f)\, d\lambda \\ & + o_\kappa(1).\end{aligned} \tag{4.26}$$

Proof. For $\eta = (\xi, \rho) \in \operatorname{supp} \hat{F}$, $\zeta \in D$ we have $|\zeta \pm c|\xi|| \geqslant 2\delta$, $|\zeta - \lambda_j(\eta)| \geqslant 2\delta$, $|\zeta - e(\xi)| \geqslant 2\delta$, and $|\zeta - m(\xi)| \geqslant 2\delta$, so that $\tau(\xi, \rho)$, $\rho + \tau(\xi, \zeta)$, and $C(\xi, \zeta)$ are regular in D, and hence so is $\mathscr{S}(\zeta)f = \mathscr{T}(\zeta)f - \mathscr{R}(\zeta)f$ by (4.5), (4.6). Thus, by Cauchy's theorem $C(N, \kappa)$ may be deformed to the disjoint union

$$\begin{aligned}\tilde{C}(N,\kappa) &= i(N,\kappa) \cup e(N,\kappa) \cup m(N,\kappa), \\ i(N,\kappa) &= i_+(N,\kappa) \cup i_-(N,\kappa), \\ e(N,\kappa) &= e_+(N,\kappa) \cup e_-(N,\kappa) \cup a_e(\kappa), \\ m(N,\kappa) &= m_+(N,\kappa) \cup m_-(N,\kappa) \cup a_m(\kappa),\end{aligned} \tag{4.27}$$

where

$$\begin{aligned} i_+(N,\kappa) &= \{\zeta = \lambda + i\kappa \colon -N < \lambda < N\}, \\ i_-(N,\kappa) &= \{\zeta = \lambda - i\kappa \colon N > \lambda \to -N\} \\ e_+(N,\kappa) &= \{\zeta = cq\lambda + iq\kappa \colon \delta < \lambda < N\}, \\ e_-(N,\kappa) &= \{\zeta = cq\lambda - iq\kappa \colon N > \lambda \to \delta\}, \\ m_+(N,\kappa) &= \{\zeta = -\varepsilon^{-1}p\lambda - i\kappa p \colon \delta < \lambda < N\}, \\ m_-(N,\kappa) &= \{\zeta = -\varepsilon^{-1}p\lambda + i\kappa p \colon N > \lambda \to \delta\},\end{aligned}$$

and $a_e(\kappa)$ and $a_m(\kappa)$ consist of line segments of length 2κ joining the end points of e_+, e_- and m_+, m_-, respectively. Since $\zeta \to \mathscr{S}(\zeta)f$ is continuous on $C(N, \kappa)$ compact

$$(g, \mathscr{G}(N,\kappa)f) = \left(g, \int_{C(N,\kappa)} \mathscr{G}(\zeta) f\, d\zeta\right) = \int_{C(N,\kappa)} (g, \mathscr{G}(\zeta)f)\, d\zeta$$
$$= \int_{\tilde{C}(N,\kappa)} (g, \mathscr{G}(\zeta)f)\, d\zeta. \tag{4.28}$$

Since $(g, \mathscr{G}(\zeta)f)$ is regular in D,

$$\begin{aligned}
\int_{i(N,\kappa)} (g, \mathscr{G}(\zeta)f)\, d\zeta &= \int_{-N}^{-\delta} (g, \mathscr{G}(\lambda + i\kappa)f)\, d\lambda + \int_{-\delta}^{-N} (g, \mathscr{G}(\lambda - i\kappa)f)\, d\lambda \\
&\quad + \int_{-\kappa}^{\kappa} (g, \mathscr{G}(-\delta + it)i\, dt + \int_{\delta}^{N} (g, \mathscr{G}(\lambda + i\kappa)f)\, d\lambda \\
&\quad + \int_{N}^{\delta} (g, \mathscr{G}(\lambda - i\kappa)f)\, d\lambda + \int_{-\kappa}^{\kappa} (g, \mathscr{G}(\delta + it)f)i\, dt \\
&\quad + \oint_{|\zeta| = \delta} (g, \mathscr{G}(\zeta)f)\, d\zeta \\
&= \int_{-N}^{-\delta} (g, [\mathscr{G}(\lambda + i\kappa) - \mathscr{G}(\lambda - i\kappa)]f)\, d\lambda \\
&\quad + \int_{\delta}^{N} (g, [\mathscr{G}(\lambda + i\kappa) - \mathscr{G}(\lambda - i\kappa)]f)\, d\lambda \\
&\quad - 2\pi i \operatorname*{Res}_{\zeta=0}(g, \mathscr{G}(\zeta)f) + o_\kappa(1) \qquad (4.29) \\
&= 2i\kappa \int_{-N}^{-\delta} (\mathscr{G}^*(\lambda \pm i\kappa)g, \mathscr{G}(\lambda \mp i\kappa)f)\, d\lambda \\
&\quad + 2i\kappa \int_{\delta}^{N} (\mathscr{G}^*(\lambda \pm i\kappa)g, \mathscr{G}(\lambda \mp i\kappa)g)\, d\lambda \\
&\quad - 2\pi i \operatorname*{Res}_{\zeta=0}(g, \mathscr{G}(\zeta)f) + o_\kappa(1)
\end{aligned}$$

by the resolvent identity. Next,

$$\begin{aligned}
\int_{e(N,\kappa)} (g, \mathscr{G}(\zeta)f)\, d\zeta &= cq \int_{\delta}^{N} (g, \mathscr{G}(cq\lambda + iq\kappa)f)\, d\lambda \\
&\quad + cq \int_{N}^{\delta} (g, \mathscr{G}(cq\lambda - iq\kappa)f)\, d\lambda + o_\kappa(1) \qquad (4.30) \\
&= 2icq^2\kappa \int_{\delta}^{N} (\mathscr{G}^*(e_\pm(\lambda))g, \mathscr{G}(e_\mp(\lambda))f) d\lambda + o_\kappa(1).
\end{aligned}$$

Finally,

$$\begin{aligned}
\int_{m(N,\kappa)} (g, \mathscr{G}(\zeta) f)\, d\zeta &= -\varepsilon^{-1} p \int_{\delta}^{N} (g, \mathscr{G}(-\varepsilon^{-1} p\lambda - i\kappa p) f)\, d\lambda \\
&\quad - \varepsilon^{-1} p \int_{N}^{\delta} (-\varepsilon^{-1} p\lambda + i\kappa p) f)\, d\lambda + o_{\kappa}(1) \\
&= -\varepsilon^{-1} p \int_{\delta}^{N} (g, [\mathscr{G}(-\varepsilon^{-1} p\lambda - i\kappa p) \\
&\quad - \mathscr{G}(-\varepsilon^{-1} p\lambda + i\kappa p)] f)\, d\lambda + o_{\kappa}(1) \\
&= 2i\kappa\varepsilon^{-1} p^2 \int_{\delta}^{N} (\mathscr{G}^*(m_{\pm}(\lambda)) g, \mathscr{G}(m_{\mp}(\lambda)) f)\, d\lambda + o_{\kappa}(1).
\end{aligned} \tag{4.31}$$

Combining (4.27)–(4.31), we obtain (4.26).

LEMMA 4.6. *Let $\chi_+(x) \equiv \chi_{R_+}(x_3)$ be the characteristic function of R^3_+, and for $f \in \mathscr{H}$ define*

$$\begin{aligned}
f_a(x) &= f(x', x_3 - a), \\
\tilde{f}_a(\xi; \zeta) &= (2\pi)^{-1/2} \int_0^{\infty} \exp(i\tau(\xi, \zeta)\, x_3)\, \Phi_2 f_a(\xi, x_3)\, dx_3.
\end{aligned} \tag{4.32}$$

Then with $\Psi(x, \eta; \zeta)$, $\Psi'(x, \eta; \zeta)$ of (3.31), (3.32)

$$\begin{aligned}
\Psi(\zeta) f(\eta) &\equiv \exp(-ia\rho)\, \Phi \chi_a \mathscr{G}^*(\zeta) f(\eta) \\
&= \int_{R^3_{-a}} {}^{\mathrm{t}}\Psi(x, \eta; \zeta)\, Ef(x)\, dx \\
&= \sum_{j=-1}^{1} [\lambda_j(\eta) - \bar{\zeta}]^{-1} P_j(\eta)\, \Phi \chi_+ f_a(\eta) - E^{-1}\, {}^{\mathrm{t}}[EM(\eta; \zeta)]\, \tilde{f}_a(\xi; \bar{\zeta}),
\end{aligned} \tag{4.33}$$

$$\begin{aligned}
\Psi'(\zeta) f(\eta) &\equiv \exp(-ia\rho)\, \Phi \chi_a \mathscr{G}(\bar{\zeta}) f(\eta) \\
&= \int_{R^3_{-a}} {}^{\mathrm{t}}\Psi'(x, \eta; \zeta)\, Ef(x)\, dx \\
&= \sum_{j=-1}^{1} [\lambda_j(\eta) - \bar{\zeta}]^{-1} P_j(\eta)\, \Phi \chi_+ f_a(\eta) - E^{-1}\, {}^{\mathrm{t}}[EM'(\eta; \zeta)]\, \tilde{f}_a(\xi; \bar{\zeta}).
\end{aligned} \tag{4.34}$$

Proof. Let $\psi \in \mathscr{K}_3$. By the Parseval equality for the Fourier transform

$$\begin{aligned}\langle \Phi\psi, E\Phi\chi_a \mathscr{G}(\zeta) f\rangle &= (\psi, \chi_a \mathscr{G}(\zeta) f)_{\mathscr{K}_3} \\ &= (\chi_a \psi, \mathscr{G}(\zeta) f) = (\mathscr{G}^*(\zeta)\, \chi_a \psi, f) \\ &= (\Phi_y^* \chi_a(y_3)\, G'(\cdot, y; \bar{\zeta})(\eta)\, E^{-1} E\Phi\psi(\eta), f) \\ &= \langle E\Phi\psi, {}^{\mathrm{t}}[\Phi_y^* \chi_a(y_3)\, G'(\cdot, y; \bar{\zeta})(\eta)\, E^{-1}]\, Ef\rangle \\ &= \langle \Phi\psi, E\, {}^{\mathrm{t}}[\exp(-ia\rho)\, \Psi'(\cdot, \eta; \bar{\zeta})]\, Ef\rangle \\ &= \langle \Phi\psi, E \exp(ia\rho)\, \Psi'(\bar{\zeta}) f\rangle\end{aligned}$$

by (3.32). This gives (4.34); Eq. (4.33) is obtained in a similar way.

We are now able to express $(g, \mathscr{G}(N, \kappa) f)$ in terms of the urmodes.

COROLLARY 4.7. *Let $f \in \mathscr{H}$ have the same properties as in Lemma* 4.5, *and let $g \in \mathscr{D}(R^3_{-a}, C^6)$. Then* (4.26) *can be written*

$$\begin{aligned}&(g, \mathscr{G}(N, \kappa) f) \\ &= 2i\kappa \int_\delta^N \langle \Psi(\lambda \pm i\kappa) g, E\Psi'(\lambda \pm i\kappa) f\rangle\, d\lambda \\ &\quad + 2i\kappa \int_{-N}^{-\delta} \langle \Psi(\lambda \pm i\kappa) g, E\Psi'(\lambda \pm i\kappa) f\rangle\, d\lambda \\ &\quad + 2i\kappa c q^2 \int_\delta^N \langle \Psi(e_\pm(\lambda)) g, E\Psi'(\bar{e}_\mp(\lambda)) f\rangle\, d\lambda \\ &\quad + 2i\kappa \varepsilon^{-1} p^2 \int_\delta^N \langle \Psi(m_\pm(\lambda)) g, E\Psi'(\bar{m}_\mp(\lambda)) f\rangle\, d\lambda \\ &\quad + 2\pi i(g, \Pi_0 f) + o_\kappa(1) \\ &= 2i \int_{R^3} d\eta \left\{ \int_\delta^N \kappa\, {}^{\mathrm{t}}[\Psi(\lambda \pm i\kappa)\, g(\eta)]\, E\Psi'(\lambda \pm i\kappa)\, f(\eta)\, d\lambda \right\} \\ &\quad + 2i \int_{R^3} d\eta \left\{ \int_{-N}^{-\delta} \kappa\, {}^{\mathrm{t}}[\Psi(\lambda \pm i\kappa)\, g(\eta)]\, E\Psi'(\lambda \pm i\kappa)\, f(\eta)\, d\lambda \right\} \\ &\quad + 2i \int_{R^3} d\eta \left\{ \kappa c q^2 \int_\delta^N {}^{\mathrm{t}}[\Psi(e_\pm(\lambda))\, g(\eta)]\, E\Psi'(\bar{e}_\mp(\lambda))\, f(\eta)\, d\lambda \right\} \\ &\quad + 2i \int_{R^3} d\eta \left\{ \kappa \varepsilon^{-1} p^2 \int_\delta^N {}^{\mathrm{t}}[\Psi(m_\pm(\lambda))\, g(\eta)]\, E\Psi'(\bar{m}_\mp(\lambda))\, f(\eta)\, d\lambda \right\} \\ &\quad + 2\pi i(g, \Pi_0 f) + o_\kappa(1),\end{aligned} \tag{4.35}$$

where

$$\Pi_0 f(x) = (2\pi)^{3/2} \int_{R^3} \{\exp[ix'\xi + ix_3\rho]\, P_0(\omega)$$
$$- \exp[ix'\xi - |\xi|\,(x_3 + a) - ia\rho]\, P_0^r(\omega)\}\, \hat{F}(\eta)\, d\eta,$$

$$P_0^r(\omega) = \begin{pmatrix} {}^t(\omega_1, \omega_2, i\,|\omega'|)\omega & 0 \\ 0 & {}^t(\omega_1, \omega_2, i\,|\omega'|)\,\omega \end{pmatrix},$$
$$\omega = (\omega_1, \omega_2, \omega_3) \in S^2,$$

and $P_0(\omega)$ *is given in* (1.4). *The operator* Π_0 *defined on this particular* (*dense*) *set of* f *by* (4.36) *is a bounded-selfadjoint projection, extends by continuity to a bounded projection of* $\mathscr{H}$ *into the null space* $\mathscr{N}(\Lambda) = \mathscr{N}(\Lambda^*)$, *and* $\Pi_0 \Lambda \subset \Lambda \Pi_0$.

Proof. The first equality follows directly from (4.26) via (4.33), (4.34), and the Parseval equality for the Fourier transform. The interchange of the order of integration made in writing the second equality is justified, since $\zeta = \lambda \pm i\kappa \to \Psi'(\zeta) f$ are continuous L_2-valued functions of $\lambda \in [\delta, N]$ for any fixed $\kappa \in (0, \kappa_0]$. The expression for $\Pi_0 f$ is obtained by computing the residue at zero in (4.26) from (1.7), (3.10), and (3.25). It is a straightforward check that $\Lambda(D) \exp(-|\xi|\, x_3 + ix'\xi)\, P_0^r(\omega) = 0$, $B\Pi_0 f(x', -a) = B'\Pi_0 f(x', -a) = 0$, and $\Pi_0 f \in \mathscr{H}$, so that $\Pi_0 f \in \mathscr{N}(\Lambda) = \mathscr{N}(\Lambda^*)$. The verification that Π_0 is a bounded-selfadjoint projection is somewhat lengthy and is given in Appendix A. We show that Π_0 reduces Λ. Let $f \in \mathscr{N}(\Lambda)$, let $\phi \in \mathscr{D}(\mathring{\Lambda}')$, and let $\{f_n\} \subset \mathscr{H}$ be a sequence of functions converging to f in $\mathscr{H}$ such that each f_n has the property of the function f of Lemma 4.5. Then $\Pi_0 f_n$ is smooth, in $\mathscr{H}$, and $B\Pi_0 f_n(x', -a) = 0$; hence, on integrating by parts, we see that $(\Pi_0 f_n, \Lambda\phi) = 0$ by (2.4). Therefore, since Π_0 is continuous, $(\Pi_0 f, \Lambda\phi) = \lim(\Pi_0 f_n, \Lambda\phi) = 0$; now this holds, in particular, for $\phi \in \mathscr{D}(R^3_{-a}, C^6)$, so $\Lambda\Pi_0 f \in \mathscr{H}$, and $\Pi_0 f$ satisfies (2.12). Hence, $\pi_0 f \in \mathscr{M}(\Lambda)$, and $\Lambda\Pi_0 f = \Pi_0 \Lambda f$.

The next step in obtaining the Parseval identity for Λ, Λ^* is to pass to the limit $\kappa \downarrow 0$ in (4.35). In order to formulate the result, we must first compute the generalized eigenfunctions of Λ, Λ^* from (3.31) and (3.32).

In computing the plane-wave modes $\Psi_j(x, \eta)_\pm \equiv [(\zeta - \lambda_j(\eta)\ \ \Psi(x, \eta; \zeta)]\ (\lambda_j(\eta) \pm i0)$, $\Psi_j'(x, \eta)_\pm = [(\zeta - \lambda_j(\eta))\ \ \Psi'(x, \eta; \zeta)]\ (\lambda_j(\eta) \pm i0)$, $j = \pm 1$, it is convenient to first observe the following facts (here, $R_{\pm j}$ is $R_\pm = \{x \in R\colon \pm x > 0\}$ for $j = 1$ and $R_\mp$ for $j = -1$);

$$\chi_{R_{\mp j}}(\rho)\, \tau(\xi, \lambda_j(\eta) \pm i0) = \pm j\,|\rho|\, \chi_{R_{\mp j}}(\rho) = -\rho\chi_{R_{\mp j}}(\rho)$$
$$= -|\eta|\, \omega_3 \chi_{R_{\mp j}}(\rho) = \pm j\,|\eta|\, \sqrt{(1 - |\omega'|^2)},$$

$$[(\lambda_j(\eta) - \zeta)/(\rho - \tau)](\lambda_j(\eta) \pm i0) = jc\rho\,|\eta|^{-1}\, \chi_{R_{\pm j}}(\rho), \tag{4.37}$$

$$[(\lambda_j(\eta) - \zeta)/(\rho + \tau)](\lambda_j(\eta) \pm i0) = jc\rho\,|\eta|^{-1}\, \chi_{R_{\mp j}}(\rho).$$

For $\omega = (\omega', \omega_3) \in S^2$ we define (see (3.12))

$$\begin{aligned}
&C(\omega) = \Delta(\omega)^{-1}\,[\varepsilon c\omega_3(\alpha^2 - \mu\varepsilon^{-1})\,Q_a + \alpha C(\omega')],\\
&C'(\omega) = \Delta'(\omega)^{-1}[\varepsilon c\omega_3(\mu\varepsilon^{-1} - \bar{\alpha}^2)\,Q_a + \bar{\alpha} C(\omega')],\\
&C_j(\omega) = C(\omega', j\omega_3), \qquad C'_j(\omega) = C'(\omega', j\omega_3), \qquad j = \pm 1,\\
&\Delta(\omega) = (\mu c - \alpha\omega_3)(\alpha\varepsilon c - \omega_3), \qquad \Delta'(\omega) = (\mu c + \bar{\alpha}\omega_3)(\bar{\alpha}\varepsilon c + \omega_3).
\end{aligned} \tag{4.38}$$

We observe that

$${}^{t}C_j(\omega) = C'_j(\tilde{\omega}), \quad C_j(\omega)\,C_j(\tilde{\omega}) = C_j(\tilde{\omega})\,C_j(\omega) = I, \quad \tilde{\omega} = (\omega', -\omega_3). \tag{4.39}$$

Passing to the limits $\zeta \to \lambda_j(\eta) \pm i0$ in (3.12), we obtain

$$\begin{aligned}
C_\pm(\omega') &\equiv C(\xi, \lambda_j(\eta) \pm i0) \;= C(\omega', \mp\sqrt{(1 - |\omega'|^2)}) \;= \chi_{R_{\mp j}}(\rho)\,C_j(\omega),\\
C'_\pm(\omega') &\equiv C'(\xi, \lambda_j(\eta) \pm i0) = C'(\omega', \mp\sqrt{(1 - |\omega'|^2)}) = \chi_{R_{\mp j}}(\rho)\,C'_j(\omega).
\end{aligned} \tag{4.40}$$

Passing to the same limits in (3.21) for $\rho \in R_{\mp j}$, $j = \pm 1$, gives the paramutation relations: with $P_j(\omega)$ of (1.4) and $j, j' = \pm 1$

$$C_{j'}(\omega)\,P_j(\omega) = P_j(\tilde{\omega})\,C_{j'}(\omega), \qquad C'_{j'}(\omega)\,P_j(\omega) = P_j(\tilde{\omega})\,C'_{j'}(\omega). \tag{4.41}$$

Now, observing (4.37), (4.40) and passing to the limits $\zeta \to \lambda_j(\eta) \pm i0$ in (3.31) and (3.32), we obtain the respective plane-wave modes for A, A^*: for $j = \pm 1$

$$\begin{aligned}
\Psi_j(x, \eta)_\pm &= (2\pi)^{-3/2}\,\chi_{R_{\mp j}}(\rho)\,e^{ix'\xi}\{e^{i\rho(x_3 + a)}P_j(\omega)\\
&\quad - e^{-i\rho(x_3 + a)}C_j(\omega)\,P_j(\omega)\}\,E^{-1}\\
&= (2\pi)^{-3/2}\,\chi_{R_{\mp j}}(\rho)\,e^{ix'\xi}\{e^{i\rho(x_3 + a)}P_j(\omega)\\
&\quad - e^{-i\rho(x_3 + a)}P_j(\tilde{\omega})\,C_j(\omega)\}\,E^{-1},\\
\Psi'_j(x, \eta)_\pm &= (2\pi)^{-3/2}\,\chi_{R_{\mp j}}(\rho)\,e^{ix'\xi}\{e^{i\rho(x_3 + a)}P_j(\omega)\\
&\quad - e^{-i\rho(x_3 + a)}C'_j(\omega)\,P_j(\omega)\}\,E^{-1},\\
&= (2\pi)^{-3/2}\,\chi_{R_{\mp j}}(\rho)\,e^{ix'\xi}\{e^{i\rho(x_3 + a)}P_j(\omega)\\
&\quad - e^{-i\rho(x_3 + a)}P_j(\tilde{\omega})\,C'_j(\omega)\}\,E^{-1},
\end{aligned} \tag{4.42}$$

where the two forms of Ψ_j, Ψ'_j follow from the paramutation relations (4.41).

From the first forms of Ψ_j, Ψ'_j it is readily verified that

$$B\Psi_j(x', -a)_\pm = 0, \qquad B'\Psi'_j(x', -a)_\pm = 0, \tag{4.43}$$

while from the second form it is evident that

$$A(D)\,\Psi_j(x,\eta)_\pm = \lambda_j(\eta)\,\Psi_j(x,\eta)_\pm, \quad A(D)\,\Psi'_j(x,\eta)_\pm = \lambda_j(\eta)\,\Psi'_j(x,\eta)_\pm, \tag{4.44}$$

The Ψ_j, Ψ'_j are thus matrix-valued generalized eigenfunctions of A, A^*. It is also evident from the second forms of (4.42) that if $c(x,\eta) = \binom{c^1(x,\eta)}{c^2(x,\eta)}$ is any column of Ψ_j, Ψ'_j, then $\operatorname{div}_x c^1(x,\eta) = \operatorname{div}_x c^2(x,\eta) = 0$. Hence, by Corollary 2.12 superpositions of $\Psi_j(x,\eta)$, $\Psi'_j(x,\eta)$ on η which are square summable on $x \in R^3_{-a}$ lie in $\tilde{\mathscr{H}}$, the complement in $\mathscr{H}$ of the null space $\mathscr{N}(A) = \mathscr{N}(A^*)$.

For $g \in \mathscr{D}(R^3_{-a}, C^6)$ let $g_a(x) = g(x', x_3 - a)$ and define (recall χ_+ is the characteristic function of R^3_+)

$$\begin{aligned}\Psi_j\, g(\eta) &\equiv \int_{R^3_{-a}} {}^t\Psi(x,\eta)_\pm\, Eg(x)\, dx \\ &= \chi_{R_{\mp j}}(\rho)\, P_j(\omega)\{\Phi_3\chi_+\, g_a(\eta) - C'_j(\tilde{\omega})\,\Phi_3\chi_+\, g_a(\tilde{\eta})\},\end{aligned} \tag{4.45}$$

$$\begin{aligned}\Psi'_j\, g(\eta) &\equiv \int_{R^3_{-a}} {}^t\Psi'_j(x,\eta)_\pm\, Eg(x)\, dx \\ &= \chi_{R_{\mp j}}(\rho)\, P_j(\omega)\{\Phi_3\chi_+\, g_a(\eta) - C_j(\tilde{\omega})\,\Phi_3\chi_+\, g_a(\tilde{\eta})\},\end{aligned} \tag{4.46}$$

$$\tilde{\eta} = (\xi, -\rho), \quad j = \pm 1,$$

where in writing out the second lines in (4.45), (4.46) we have used relation (4.39).

LEMMA 4.8. *Ψ_j, Ψ'_j extend to bounded maps $\mathscr{H} \to \mathscr{K}_3$.*

Proof. We first observe that the denominators $\Delta(\omega', \mp\sqrt{(1-|\omega'|^2)})$, $\Delta'(\omega', \mp\sqrt{(1-|\omega'|^2)})$ of the reflection coefficients $\chi_{R_{\mp j}}(\rho)\, C'_j(\tilde{\omega})$, $\chi_{R_{\mp j}}(\rho)\, C_j(\tilde{\omega})$ are nonzero for $|\omega'| \leqslant 1$ by (4.38) and are, hence, bounded away from zero for $|\omega'| \leqslant 1$. Therefore,

$$\begin{aligned}|\Psi_j\, g(\eta)|^2 &\leqslant 2\chi_{R_{\mp j}}(\rho)\{|P_j(\omega)\,\Phi_3\chi_+\, g_a(\eta)|^2 + |C'_j(\tilde{\omega})\,\Phi_3\chi_+\, g_a(\tilde{\eta})|^2\} \\ &\leqslant \text{const}\,\chi_{R_{\mp j}}(\rho)\{|\Phi_3\chi_+\, g_a(\eta)|^2 + |\Phi_3\chi_+\, g_a(\tilde{\eta})|^2\},\end{aligned}$$

and, hence, $|\Psi_j\, g|_{\mathscr{K}_3} \leqslant \text{const}\,|g|_{\mathscr{H}}$. The assertion for Ψ'_j is verified in the same way.

To record the surface-wave modes corresponding to the roots $e(\xi) = cq\,|\xi|$,

$m(\xi) = -\varepsilon^{-1}p|\xi|$ of $\Delta(\xi, \zeta)$ (see Lemma 3.1), using (3.23), we define the vectors

$$\begin{aligned}
\mathscr{E}(\xi) &= {}^{\mathrm{t}}(\xi_2 e(\xi), -\xi_1 e(\xi), 0, -\alpha^{-1}\xi_1 e(\xi), -\alpha^{-1}\xi_2 e(\xi), -\mu^{-1}|\xi|^2) \\
&= {}^{\mathrm{t}}(-\xi_2 \alpha\mu^{-1}\tau_e, \xi_1 \alpha\mu^{-1}\tau_e, 0, \mu^{-1}\xi_1\tau_e, \mu^{-1}\xi_2\tau_e, -\mu^{-1}|\xi|^2), \\
\mathscr{E}'(\xi) &= {}^{\mathrm{t}}(\xi_2 e(\xi), -\xi_1 e(\xi), 0, \alpha^{-1}\xi_1 e(\xi), \alpha^{-1}\xi_2 e(\xi), -\mu^{-1}|\xi|^2) \\
&= {}^{\mathrm{t}}(-\xi_2 \alpha\mu^{-1}\tau_e, \xi_1 \alpha\mu^{-1}\tau_e, 0, -\mu^{-1}\xi_1\tau_e, -\mu^{-1}\xi_2\tau_e, -\mu^{-1}|\xi|^2), \\
\mathscr{M}(\xi) &= {}^{\mathrm{t}}(\alpha\xi_1 m(\xi), \alpha\xi_2 m(\xi), \varepsilon^{-1}|\xi|^2, \xi_2 m(\xi), -\xi_1 m(\xi), 0) \\
&= {}^{\mathrm{t}}(-\varepsilon^{-1}\xi_1\tau_m, -\varepsilon^{-1}\xi_2\tau_m, \varepsilon^{-1}|\xi|^2, -\alpha^{-1}\varepsilon^{-1}\xi_2\tau_m, \alpha^{-1}\varepsilon^{-1}\xi_1\tau_m, 0), \\
\mathscr{M}'(\xi) &= {}^{\mathrm{t}}(\alpha\xi_1 m(\xi), \alpha\xi_2 m(\xi), -\varepsilon^{-1}|\xi|^2, -\xi_2 m(\xi), \xi_1 m(\xi), 0) \\
&= {}^{\mathrm{t}}(-\varepsilon^{-1}\xi_1\tau_m, -\varepsilon^{-1}\xi_2\tau_m, -\varepsilon^{-1}|\xi|^2, \varepsilon^{-1}\alpha^{-1}\xi_2\tau_m, -\varepsilon^{-1}\alpha^{-1}\xi_1\tau_m, 0).
\end{aligned} \tag{4.47}$$

We observe that

$$\begin{aligned}
{}^{\mathrm{t}}\mathscr{E}'E\mathscr{E} &= 2\mu^{-1}|\xi|^4, & {}^{\mathrm{t}}\mathscr{M}'E\mathscr{M} &= -2\varepsilon^{-1}|\xi|^4, \\
{}^{\mathrm{t}}\mathscr{E}'E\mathscr{M} &= 0, & {}^{\mathrm{t}}\mathscr{M}'E\mathscr{E} &= 0.
\end{aligned} \tag{4.48}$$

Now setting

$$\beta_E = (iq/2\alpha c\varepsilon)^{1/2}, \qquad \beta_M = (-ip\alpha/2)^{1/2} \tag{4.49}$$

for some fixed choice of the branch of $(\cdot)^{1/2}$, say $\mathrm{Im}(\cdot)^{1/2} \geqslant 0$, from (3.31), (3.32) we compute the limits

$$\begin{aligned}
\Sigma_E(x, \xi) &\equiv (2\pi)^{1/2}\beta_E(-2i\beta_E^2)^{-1}|\xi|^{-1/2}[\rho + \tau_e(\xi)] \lim_{\zeta \to e(\xi)} [\zeta - e(\xi)]\, \Psi(x, \eta; \zeta) \\
&= (2\pi)^{-1}\mu\beta_E|\xi|^{-7/2}\exp[ix'\xi + i\tau_e(\xi)(x_3 + a)]\, \mathscr{E}\,{}^{\mathrm{t}}\mathscr{E}', \\
\Sigma_E'(x, \xi) &= -(2\pi)^{1/2}\bar{\beta}_E(-2i\beta_E^2)^{-1}|\xi|^{-1/2}[\rho - \bar{\tau}_e(\xi)] \lim_{\bar{\zeta} \to \bar{e}(\xi)} [\bar{\zeta} - \bar{e}(\xi)]\, \Psi'(x, \eta; \bar{\zeta}) \\
&= (2\pi)^{-1}\mu\bar{\beta}_E|\xi|^{-7/2}\exp[ix'\xi - i\bar{\tau}_e(x_3 + a)]\, \bar{\mathscr{E}}'\,{}^{\mathrm{t}}\bar{\mathscr{E}} \\
\Sigma_M(x, \xi) &= (2\pi)^{1/2}\beta_M(2i\beta_M^2)^{-1}|\xi|^{-1/2}[\rho + \tau_m(\xi)] \lim_{\zeta \to m(\xi)} [\tau - m(\xi)]\, \Psi(x, \eta; \zeta) \\
&= (2\pi)^{-1}\varepsilon\beta_M|\xi|^{-7/2}\exp[ix'\xi + i\tau_m(x_3 + a)]\, \mathscr{M}\,{}^{\mathrm{t}}\mathscr{M}', \\
\Sigma_M'(x, \xi) &= -(2\pi)^{1/2}\bar{\beta}_M(\overline{-2i\beta_M^2})^{-1}|\xi|^{-1/2}[\rho - \bar{\tau}_m(\xi)] \lim_{\bar{\zeta} \to \bar{m}(\zeta)} [\bar{\zeta} - \bar{m}(\xi)]\, \Psi'(x, \eta; \bar{\zeta}) \\
&= (2\pi)^{-1}\varepsilon\bar{\beta}_M|\xi|^{-7/2}\exp[ix'\xi - i\bar{\tau}_m(x_3 + a)]\, \bar{\mathscr{M}}'\,{}^{\mathrm{t}}\bar{\mathscr{M}}.
\end{aligned} \tag{4.50}$$

It is a straightforward computation to show that

$$B\Sigma_E(x', -a, \xi) = B\Sigma_M(x', -a, \xi) = B'\Sigma_E'(x', -a, \xi) = B'\Sigma_M'(x', -a, \xi) = 0,$$
$$\Lambda(D)\,\Sigma_E(x, E) = e(\xi)\,\Sigma_E(x, \xi), \qquad \Lambda(D)\,\Sigma_E'(x, \xi) = \bar{e}(\xi)\,\Sigma_E'(x, \xi), \tag{4.51}$$
$$\Lambda(D)\,\Sigma_M(x, \xi) = m(\xi)\,\Sigma_M(x, \xi), \qquad \Lambda(D)\,\Sigma_M'(x, \xi) = \bar{m}(\xi)\,\Sigma_M'(x, \xi).$$

Furthermore, the columns of the Σ's are divergence free, and so superpositions of them on ξ which are square summable on $x \in R^3_{-a}$ lie in $\tilde{\mathscr{H}}$.

Lemma 4.9. *The operators Σ_E, Σ_E', Σ_M, Σ_M' defined on $f \in \mathscr{D}(R^3_{-a}, C^6)$ by affixing the corresponding subscript and eventual prime to*

$$\Sigma f(\xi) = \int_{R^3_{-a}} {}^{\bar{t}}\Sigma(x, \xi)\, Ef(x)\, dx \tag{4.52}$$

extend to bounded maps $\mathscr{H} \to \mathscr{K}_2$.

Proof. Consider for example

$$\Sigma_E f(\xi) = \mu\bar{\beta}_E\, |E|^{-7/2}\, \bar{\mathscr{E}}'(\xi)\, {}^{t}\bar{\mathscr{E}}(\xi) E \int_{-a}^{\infty} \exp[-i\bar{\tau}_e(\xi)(x_3 + a)]\, \Phi_2 f(\xi, x_3)\, dx_3.$$

Now $\operatorname{Im} \tau_e = (q_1\alpha_2 - q_2\alpha_1)\,|\alpha|^{-2}\,\mu c\,|\xi| \equiv b\,|\xi|$, $b > 0$, (see (3.22), and hence

$$\left| \int_{-a}^{\infty} \exp[-i\bar{\tau}_e(\xi)(x_3 + a)]\, \Phi_2 f(\xi, x_3)\, dx_3 \right|^2$$
$$< 2^{-1} b^{-1}\, |\xi|^{-1} \int_{-a}^{\infty} |\Phi_2 f(\xi, x_3)|^2\, dx_3.$$

Since $[\bar{\mathscr{E}}'(\xi)\, {}^{t}\bar{\mathscr{E}}(\xi)]_{ij} \sim |\xi|^4$, $i, j = 1, \ldots, 6$,

$$|\Sigma_E f(\xi)|^2 < \text{const} \int_{-a}^{\infty} |\Phi_2 f(\xi, x_3)|^2\, dx_3,$$

and, hence, integrating on $\xi \in R^2$ gives $|\Sigma_E f|_{\mathscr{K}_2} < \text{const}\, |f|_{\mathscr{H}}$. The demonstration for the other Σ's is exactly the same.

We now return to Corollary 4.7 and pass to the limit $\kappa \downarrow 0$ in (4.35). The proof that the result is as stated below is quite technical and is presented in Appendix B.

Lemma 4.10. *Let $\mathscr{H} \ni f = \chi_a F$ be the restriction to R^3_{-a} of a function $F \in \mathscr{K}_3$ such that $\hat{F} \in \mathscr{D}(R^3, C^6)$ and $\{|\xi| = 0\} \cap \operatorname{supp} \hat{F} = \varnothing$, and let*

$g \in \mathscr{D}(R^3_{-a}, C^6)$. *Then with the maps* (4.45), (4.46), (4.52), *and the projection of* (4.36)

$$\begin{aligned}(2i\pi)^{-1} \lim_{\kappa \downarrow 0}(g, \mathscr{G}(N,\kappa)f) = (g, \Pi_0 f) &+ \int_{\{\lambda_1(\eta)\in(\delta,N)\}} {}^{\bar{t}}[\Psi_1 g(\eta)]\, E\Psi'_1 f(\eta)\, d\eta \\ &+ \int_{\{\lambda_{-1}(\eta)\in(\delta,N)\}} {}^{\bar{t}}[\Psi_{-1} g(\eta)]\, E\Psi'_{-1} f(\eta)\, d\eta \\ &+ \int_{\{|\xi|\in(\delta,N)\}} {}^{\bar{t}}[\Sigma_E\, g(\xi)]\, E\Sigma'_E f(\xi)\, d\xi \\ &+ \int_{\{|\xi|\in(\delta,N)\}} {}^{\bar{t}}[\Sigma_M\, g(\xi)]\, E\Sigma'_M f(\xi)\, d\xi.\end{aligned} \tag{4.53}$$

Combining Lemma 4.10 *and Theorem* 4.4, *we now obtain the Parseval equality for* Λ, Λ^*.

THEOREM 4.11. *Let* Λ, Λ^* *be the maximal-dissipative operators in* $\mathscr{H}$ *engendered by the operator* $\Lambda(D)$ *of* (0.2) *and the boundary conditions* (2.1), (2.1′) *with* $\operatorname{Re} \alpha = \alpha_1 > 0$, $\operatorname{Im} \alpha_2 \neq 0$. *Then for any* $g, f \in \mathscr{H}$ *with the maps* (4.45), (4.46), (4.52), *and the projection of* (4.36)

$$(g,f) = \sum_{j=\pm 1} (\Psi_j\, g, \Psi'_j f)_{\mathscr{K}_3} + \sum_{S=E,M} (\Sigma_S\, g, \Sigma'_S f)_{\mathscr{K}_2} + (g, \Pi_0 f). \tag{4.54}$$

Proof. For the special g and f of the hypotheses of Lemma 4.10 equality (4.54) follows from (4.25) and (4.53) by letting $N \uparrow \infty$ and $\delta \downarrow 0$, since the integrands in (4.53) are summable functions by Lemma 4.8 and 4.9. Since by the same lemmas, the mappings appearing on the right side of (4.54) are bounded, the result now follows by approximating any g and f in $\mathscr{H}$ by sequences of functions having the special properties listed in the hypotheses of the lemma.

We now proceed to decompose the space $\mathscr{H}$ into subspaces on each of which the semigroups $S(t)$ and $S^*(t)$ have simple representations.

LEMMA 4.12. *If* $f \in \mathscr{K}_3$, $g \in \mathscr{K}_2$ *are compactly supported the adjoints of* Ψ_j, Σ_S, Ψ'_j, Σ'_S, $S = E$, M, $j = \pm 1$, *are given by affixing the appropriate subscript and eventual prime to*

$$\begin{aligned}\Psi^* f(x) &= \int_{R^3} \Psi(x,\eta)\, Ef(\eta)\, d\eta, \\ \Sigma^* g(x) &= \int_{R^2} \Sigma(x,\xi)\, Eg(\xi)\, d\xi.\end{aligned} \tag{4.55}$$

Formulas (4.55) *hold for any* $f \in \mathscr{H}_3$, $g \in \mathscr{H}_2$ *in the sense of the convergence in* $\mathscr{H}$ *of* $\Sigma^* g_N \to \Sigma^* g$, $\Psi^* f_N \to \Psi^* f$, *where* $f_N \to f$, $g_N \to g$ *are compactly supported functions converging in* $\mathscr{H}_3$, $\mathscr{H}_2$ *to* f *and* g. *All these adjoint maps have range in* $\tilde{\mathscr{H}}$.

Proof. Let $f \in \mathscr{H}_3$ have compact support, and let $h \in \mathscr{D}(R^3_{-a}, C^6)$, then

$$(h, \Psi^* f) = (\Psi h, f)_{\mathscr{H}_3} = \int_{R^3} \left\{ \int_{R^3_{-a}} {}^{\bar{t}}h(x)\, E\Psi(x, \eta)\, dx \right\} Ef(\eta)\, d\eta.$$

Now ${}^{\bar{t}}h(x)\, E\Psi(x, \eta)$ is an L_1-function with respect to x with L_1-norm depending continuously on η for $\rho \neq 0$ and boundedly on $\eta \in \operatorname{supp} f$. The order of integration may thus be interchanged. The proof for Σ^* is the same. The remaining assertions are obvious.

LEMMA 4.13. *If* $f \in \mathscr{D}(\Lambda)$ *and* $g \in \mathscr{D}(\Lambda^*)$ *are smooth and rapidly decreasing, then the following equalities hold pointwise*:

$$\begin{aligned} &\Psi_j' \Lambda f(\eta) = \lambda_j(\eta)\, \Psi_j' f(\eta), && \Psi_j \Lambda^* g(\eta) = \lambda_j(\eta)\, \Psi_j\, g(\eta), && j = \pm 1, \\ &\Sigma_E' \Lambda f(\xi) = e(\xi)\, \Sigma_E' f(\xi), && \Sigma_E \Lambda^* g(\xi) = \bar{e}(\xi)\, \Sigma_E\, g(\xi), && (4.56) \\ &\Sigma_M' \Lambda f(\xi) = m(\xi)\, \Sigma_M' f(\xi), && \Sigma_M \Lambda^* g(\xi) = \bar{m}(\xi)\, \Sigma_M\, g(\xi). && \end{aligned}$$

For any $f \in \mathscr{D}(\Lambda)$, $g \in \mathscr{D}(\Lambda^*)$ *these equalities hold in the* L_2 *sense.*

Proof. Define the multiplication operator M_j in $\mathscr{H}_3$ by $\mathscr{D}(M_j) = \{f \in \mathscr{H}_3 : \lambda_j(\cdot) f \in \mathscr{H}_3\}$ and for $f \in \mathscr{D}(M_j)$, $M_j f(\eta) = \lambda_j(\eta) f(\eta)$; it is clear that M_j is a closed operator (it is selfadjoint). Let $f \in \mathscr{D}(\Lambda)$ be a smooth, rapidly decreasing function. Integration by parts gives

$$\begin{aligned} \Psi_j' \Lambda f(\eta) &= \int_{R^3_{-a}} {}^{\bar{t}}\Psi_j'(x, \eta)\, E\Lambda(D) f(x)\, dx = \int_{R^3_{-a}} {}^{\bar{t}}[\Lambda(D)\, \Psi'(x, \eta)]\, Ef(x)\, dx \\ &\quad + i \int_{R^2} {}^{\bar{t}}\Psi_j'(x', -a, \eta)\, \Lambda_3 f(x', -a)\, dx' \\ &= \lambda_j(\eta)\, \Psi_j' f(\eta) = M_j \Psi_j' f(\eta) \end{aligned}$$

by (2.4), (4.43). In particular, this holds if $f \in \mathscr{D}(\mathring{\Lambda})$ (see (2.6)). Since Λ is the graph closure of Λ on $\mathscr{D}(\mathring{\Lambda})$ and Ψ_j' is bounded, if $f \in \mathscr{D}(\Lambda)$ and $\{f_n\} \subset \mathscr{D}(\mathring{\Lambda})$ converges to f in graph norm, then

$$|\Psi_j' \Lambda f - \Psi_j' \Lambda f_n|_{\mathscr{H}_3} + |\Psi_j' f - \Psi_j' f_n|_{\mathscr{H}_3} \leqslant \text{const}[|\Lambda f - \Lambda f_n| + |f - f_n|] \to 0,$$

and, hence, in the sense of $\mathscr{H}_3$

$$\Psi_j' \Lambda f = \lim_{n\to\infty} \Psi_j' \Lambda f_n = \lim_{n\to\infty} M_j \Psi_j' f_n = M_j \Psi_j' f(\eta),$$

since M_j is closed. The other relations of (4.56) follow in the same way.

LEMMA 4.14. *For Λ there are the "orthogonality" relations*

$$\begin{aligned} 0 &= \Sigma_M' \Sigma_E^* = \Sigma_E' \Sigma_M^* = \Psi_1' \Psi_{-1}^* = \Psi_{-1}' \Psi_1^*, \\ 0 &= \Psi_j' \Sigma_S^* = \Sigma_S' \Psi_j^*, \qquad j = \pm 1, \quad S = E, M. \end{aligned} \tag{4.57}$$

The "orthogonality" relations for Λ^* are the adjoints of these.

Proof. Let $f \in \mathscr{D}(R^2, C^6)$. From (4.50), (4.55)

$$\begin{aligned} \Sigma_E^* f(x) &= \int_{R^2} \Sigma_E(x, \xi)\, E f(\xi)\, d\xi \\ &= (2\pi)^{-1} \mu\beta_E \int_{R^2} |\xi|^{-7/2} \exp[ix'\xi + i\tau_e(x_3 + a)] \\ &\quad \times \mathscr{E}(\xi)\, {}^t\mathscr{E}'(\xi)\, E f(\xi)\, d\xi, \end{aligned} \tag{4.58}$$

$$\Phi_2 \Sigma_E^*(\xi, x_3) = \mu\beta_E |\xi|^{-7/2} \exp[i\tau_e(x_e + a)]\, \mathscr{E}(\xi)\, {}^t\mathscr{E}'(\xi)\, E f(\xi).$$

From (4.50), (4.52) for $g \in \mathscr{H}$

$$\Sigma_M' g(\xi) = \varepsilon\beta_M |\xi|^{-7/2} \mathscr{M}(\xi)\, {}^t\mathscr{M}'(\xi) E \int_{-a}^{\infty} \exp[i\tau_M(x_3 + a)]\, \Phi_2\, g(\xi, x_3)\, dx_3,$$

and the relation $\Sigma_M' \Sigma_E^* = 0$ thus follows directly from (4.48). The equality $0 = \Sigma_E' \Sigma_M^*$ is obtained in the same way. Now from (4.51), (4.58)

$$\Phi_2 \Lambda(D)\, \Sigma_E^* f(\xi, x_3) = \mu\beta_E |E|^{-7/2} e(\xi) \exp[i\tau_e(x_3 + a)]\, \mathscr{E}(\xi)\, {}^t\mathscr{E}'(\xi)\, E f(\xi), \tag{4.59}$$

and hence as in the proof of Lemma 4.9,

$$\int_{-a}^{\infty} |\Phi_2 \Lambda(D)\, \Sigma_E^* f(\xi, x_3)|^2\, dx_3 \leqslant \text{const}\, |\xi|^2\, |f(\xi)|^2.$$

Therefore, $\Lambda(D)\, \Sigma_E^* f \in \mathscr{H}$, and from (4.51) $B\Sigma_E^* f(x', -a) = 0$, Hence, $\Sigma_E^* f \in \mathscr{D}(\Lambda)$, and by (4.58), (4.59)

$$\Phi_2 \Lambda \Sigma_E^* f(\xi, x_3) = e(\xi)\, \Phi_2 \Sigma_E^* f(\xi, x_3).$$

Thus, by (4.46)

$$\Psi_j' \Lambda \Sigma_E^* f(\eta) = e(\xi)\, \Psi_j' \Sigma_E^* f(\eta). \tag{4.60}$$

On the other hand, $\Sigma_E^* f(x)$ is smooth and rapidly decreasing, so by (4.56)

$$\Psi_j' \Lambda \Sigma_E^* f(\eta) = \lambda_j(\eta)\, \Psi_j' \Sigma_E^* f(\eta). \tag{4.61}$$

Subtracting (4.60) from (4.61), we see that $\Psi_j' \Sigma_E^* f(\eta) = 0$. Since such f are dense in $\mathscr{H}_2$, we have $\Psi_j' \Sigma_E^* = 0$. The proof that $\Psi_j' \Sigma_E^* = 0$ is similar as are the relations $\Sigma_S' \Psi_j^* = [\Psi_j (\Sigma_S')^*]^* = 0$. To prove, e.g., $\Psi_{-1}' \Psi_1^* = 0$, we proceed as follows (see [12]). Let $\mathring{\mathscr{D}} = \{f \in \mathscr{D}(R^3, C^6) \colon \{p = 0\} \cap \operatorname{supp} f = \varnothing\}$; it is clear that $\mathring{\mathscr{D}}$ is dense in $\mathscr{H}_3$. Choose any $f \in \mathring{\mathscr{D}}$, fix an arbitrary $p \in R^3$, and set $g(\eta) = f(\eta)/c(|p| + |\eta|) \in \mathring{\mathscr{D}}$; then $\Psi_1^* g \in \mathscr{H}$ is smooth, rapidly decreasing, satisfies $B \Psi_1^* g(x', -a) = 0$ by (4.43), so is in $\mathscr{D}(\Lambda)$, and $\Lambda \Psi_1^* g(x) = \Psi_1^* c\, |\cdot|\, g(x)$. Hence,

$$\Psi_{-1}' \Lambda \Psi_1^* g(p) = \Psi_{-1}' \Psi_1^* c\, |\cdot|\, g(p). \tag{4.62}$$

On the other hand, from (4.56)

$$\Psi_{-1}' \Lambda \Psi_1^* g(p) = -c\, |p|\, \Psi_{-1}' \Psi_1^* g(p). \tag{4.63}$$

Subtracting (4.63) from (4.62), we have

$$0 = \Psi_{-1}' \Psi_1^* c(|p| + |\cdot|)\, g(p) = \Psi_{-1}' \Psi_1^* f(p).$$

Since p is an arbitrary point we have $\Psi_{-1}' \Psi_1^* f = 0$, and, since $\mathring{\mathscr{D}}$ is dense, thus also $\Psi_{-1}' \Psi_1^* = 0$. The relation $\Psi_1' \Psi_{-1}^* = 0$ is proved in the same way.

Remark. The proof of the orthogonality relation $\Psi_j \Sigma_S^* = 0$ in [12] is incorrect. It may be replaced by the argument above.

LEMMA 4.15. *The operators $\Pi_j = \Psi_j^* \Psi_j'$, $\Pi_S = \Sigma_S^* \Sigma_S'$, $j = \pm 1$, $S = E$, M, Π_0 of* (4.36), *and their adjoints are bounded projections in $\mathscr{H}$ which are "orthogonal" in the sense that*

$$\begin{aligned} 0 &= \Pi_j \Pi_{j'}, && j \neq j' = 0, \pm 1, \\ 0 &= \Pi_j \Pi_S = \Pi_S \Pi_j, && j = 0, \pm 1, \qquad S = E, M, \\ 0 &= \Pi_E \Pi_M = \Pi_M \Pi_E. \end{aligned} \tag{4.63}$$

In particular, Π_0 is the selfadjoint projection in $\mathscr{H}$ onto *the null space $\mathscr{N}(\Lambda) = \mathscr{N}(\Lambda^*)$. The remaining projections are given explicitly by*

$$\begin{aligned}
\Phi_2 \Pi_E f(\xi, x_3) &= i\mu q(ca\varepsilon)^{-1}|\xi|^{-3} \exp[i\tau_e(x_3+a)]\mathscr{E}\,{}^t\mathscr{E}'E \\
&\quad \times \int_{-a}^{\infty} \exp[i\tau_e(y_3+a)]\, \Phi_2 f(\xi, y_3)\, dy_3, \\
\Phi_2 \Pi_M f(\xi, x_3) &= i\varepsilon p a\, |\xi|^{-3} \exp[i\tau_m(x_3+a)]\mathscr{M}\,{}^t\mathscr{M}'E \\
&\quad \times \int_{-a}^{\infty} \exp[i\tau_m(y_3+a)]\, \Phi_2 f(\xi, y_3)\, dy_3, \\
\Phi_2 \Pi_E^* f(\xi, x_3) &= -i\bar{q}\mu(c\bar{a}\varepsilon)^{-1}\, |\xi|^{-3} \exp[-i\bar{\tau}_e(x_3+a)]\, \bar{\mathscr{E}}'\,{}^t\bar{\mathscr{E}}E \\
&\quad \times \int_{-a}^{\infty} \exp[-i\bar{\tau}_e(y_3+a)]\, \Phi_2 f(\xi, y_3)\, dy_3, \\
\Phi_2 \Pi_M^* f(\xi, x_3) &= -i\bar{p}\bar{a}\varepsilon\, |\xi|^{-3} \exp[-i\bar{\tau}_m(x_3+a)]\, \bar{\mathscr{M}}'\,{}^t\bar{\mathscr{M}}E \\
&\quad \times \int_{-a}^{\infty} \exp[-i\bar{\tau}_m(y_3+a)]\, \Phi_2 f(\xi, y_3)\, dy_3,
\end{aligned} \tag{4.64}$$

and

$$\begin{aligned}
(\Pi_j f)_a\,(x) &= \chi_+\, \Phi_3^* \check{f}_j(x), \qquad \check{f}_j(\eta) = P_j(\omega)[\Phi_3 \chi_+ f_a(\eta) - C_j(\tilde{\omega})\, \Phi_3 \chi_+ f_a(\tilde{\eta})] \\
(\Pi_j^* f)_a\,(x) &= \chi_+\, \Phi_3^* \check{f}_j'(x), \qquad \check{f}_j'(\eta) = P_j(\omega)[\Phi_3 \chi_+ f_a(\eta) - C_j'(\tilde{\omega})\, \Phi_3 f_a(\tilde{\eta})], \\
&\quad x \in R_+^3, \qquad f_a(x) = f(x', x_3 - a), \qquad j = \pm 1.
\end{aligned} \tag{4.65}$$

Proof. With the exception of the case $j=0$ all the assertions of (4.63) follow immediately from Lemma 4.14. Let $f \in \mathscr{N}(A)$; then $0 = \Psi_j' Af(\eta) = \lambda_j(\eta)\, \Psi_j' f(\eta) = \Sigma_E' Af(\xi) = e(\xi)\, \Sigma_E' f(\xi) = \Sigma_M' Af(\xi) = m(\xi)\, \Sigma_M' f(\xi)$, and, hence, $\Pi_j f = \Pi_S f = 0$, $j = \pm 1$, $S = E$, M, so, in particular, $\Pi_j \Pi_0 = \Pi_S \Pi_0 = 0$, $j = \pm 1$, $S = E$, M. Further, from (4.54) the Parseval identity for A, A^* can now be written

$$\begin{aligned}
I_{\mathscr{H}} &= \Pi_{-1} + \Pi_0 + \Pi_1 + \Pi_E + \Pi_M, \\
I_{\mathscr{H}} &= \Pi_{-1}^* + \Pi_0 + \Pi_1^* + \Pi_E^* + \Pi_M^*
\end{aligned} \tag{4.66}$$

and hence $f \in \mathscr{N}(A)$ implies $\Pi_0 f = f$. Π_0 is therefore the projection in $\mathscr{H}$ onto $\mathscr{N}(A)$. The relations $0 = \Pi_0 \Pi_j = (\Pi_j^* \Pi_0) = \Pi_0 \Pi_S = (\Pi_S^* \Pi_0)^*$, $j = \pm 1$, $S = E$, M, follow in the same way, using A^*, Ψ_j, and Σ_S. Relations (4.63), (4.66) now imply, e.g.,

$$\Pi_{-1} f = (\Pi_{-1} + \Pi_0 + \Pi_1 + \Pi_E + \Pi_M)\, \Pi_{-1} f = \Pi_{-1}^2 f,$$

and so Π_{-1} is a projection. That the other operators of (4.63) are projections follows in the same way. We note that in Section 5 it is also shown directly that the operators (4.64) are projections. The explicit forms of the projections (4.64) follow directly from the definitions. We derive the form of $\Pi_j, j=\pm 1$. From the definition of $\check{f}_j$, (4.39), and (4.41)

$$C_j(\tilde{\omega})\,\check{f}_j(\tilde{\eta}) = -\check{f}_j(\eta), \qquad \tilde{\eta}=(\xi,-\rho), \qquad \tilde{\omega}=(\omega',-\omega_3), \tag{4.67}$$

and, hence, from (4.42), (4.45), and (4.55) for $x_3 \in R_+$

$$\begin{aligned}
(\Pi_j f)_a &= \Psi_j^* \Psi_j' f(x', x_3 - a)\\
&= (2\pi)^{-3/2} \int_{R^3} [e^{ix\eta} I - e^{ix\tilde{\eta}} C_j(\omega)]\, \chi_{R_{\mp j}}(\rho)\, \check{f}_j(\eta)\, d\eta\\
&= (2\pi)^{-3/2} \int_{R^3} [e^{ix\eta} \chi_{R_{\mp j}}(\eta)\, \check{f}_j(\eta) - e^{ix\eta} \chi_{R_{\pm j}}(\rho)\, C_j(\tilde{\omega})\, \check{f}_j(\tilde{\eta})]\, d\eta\\
&= \Phi_3^* \check{f}_j(x).
\end{aligned}$$

The expression for Π_j^* is derived in the same way.

Remark. In the selfadjoint case the relations (4.63) state that distinct projections have orthogonal ranges, because the projections are selfadjoint. This is, of course, no longer the case here, but the relations (4.63) do imply that the ranges of distinct projections have only the zero vector in common, and hence the decompositions of $\mathscr{H}$ corresponding to (4.66) are direct-sum decompositions:

$$\mathscr{H} = \Pi_0 \mathscr{H} + \Pi_{-1} \mathscr{H} + \Pi_1 \mathscr{H} + \Pi_E \mathscr{H} + \Pi_M \mathscr{H}, \tag{4.67}$$

$$\mathscr{H} = \Pi_0 \mathscr{H} + \Pi_{-1}^* \mathscr{H} + \Pi_1^* \mathscr{H} + \Pi_E^* \mathscr{H} + \Pi_M^* \mathscr{H}. \tag{4.67*}$$

THEOREM 4.16. *The direct-sum decompositions* (4.67), (4.67*) *reduce* $S(t)$, $S^*(t)$, *and for* $f \in \mathscr{H}$ *these semigroups have the respective representations*

$$\begin{aligned}
S(t)f &= \Pi_0 f + S(t)\,\Pi_1 f + S(t)\,\Pi_{-1} f + S(t)\,\Pi_E f + S(t)\,\Pi_M f\\
&= \Pi_0 f + \Psi_1^* \exp(-ic\,|\cdot|\,t)\,\Psi_1' f + \Psi_{-1}^* \exp(ic\,|\cdot|\,t)\,\Psi_{-1}' f\\
&\quad + \Sigma_E^* \exp(-ie(\cdot)t)\,\Sigma_E' f + \Sigma_M^* \exp(-im(\cdot)t)\,\Sigma_M' f,
\end{aligned} \tag{4.68}$$

$$\begin{aligned}
S^*(t)f &= \Pi_0 f + S^*(t)\,\Pi_1^* f + S^*(t)\,\Pi_{-1}^* f + S^*(t)\,\Pi_E^* f + S^*(t)\,\Pi_M^* f\\
&= \Pi_0 f + (\Psi_1')^* \exp(ic\,|\cdot|\,t)\,\Psi_1 f + (\Psi_{-1}')^* \exp(-ic\,|\cdot|\,t)\,\Psi_{-1} f\\
&\quad + (\Sigma_E')^* \exp(i\bar{e}(\cdot)t)\,\Sigma_E f + (\Sigma_M')^* \exp(i\bar{m}(\cdot)t)\,\Sigma_M f.
\end{aligned} \tag{4.68*}$$

If $f \in \mathscr{D}(A)$, *then* (4.68) *is a representation of the solution in* $\mathscr{H}$ *of problem* (0.2)–(0.4).

Proof. We call the right side of (4.68) $S_1(t)$ and show that $S_1(t)=S(t)$. First, it is clear that for each t $S_1(t)$ is a bounded operator in $\mathscr{H}$ as a sum of compositions of bounded mappings. Next, if $f\in\mathscr{D}(R^3_{-a}, \mathrm{C}^6)$, then $\Pi_0 f\in\mathscr{D}(\Lambda)$ by Corollary 4.7, and $\Psi'_j f$, $\Sigma'_S f$, $j=\pm 1$, $S=E, M$, are rapidly decreasing functions. It is then clear that $\Psi^*_j \exp(-ijc\,|\cdot|\,t)\,\Psi'_j f$, $\Sigma^*_E \exp(-ie(\cdot)t)\,\Sigma'_E f$, and $\Sigma^*_M \exp(-im(\cdot)t)\,\Sigma'_M f$ are smooth, belong to $\mathscr{D}(\Lambda)$, $i\partial_t S_1(t)f=\Lambda S_1(t)f$, and $S_1(0)f=f$ by (4.67). Now the unique solution in $\mathscr{H}$ of this Cauchy problem is $S(t)f=\exp(-it\Lambda)f$ [7]. Hence, $S(t)=S_1(t)$ on a dense set in $\mathscr{H}$, and, since they are both bounded, they coincide everywhere. The proof that $S^*(t)$ has the representation (4.68*) is similar. To show that the decomposition (4.67) reduces $S(t)$, let, e.g., $f\in\Pi_E\mathscr{H}$; then by Lemmas 4.14, 4.15, and (4.68) $S(t)f=\Sigma^*_E \exp(-ie(\cdot)t)\,\Sigma'_E f$, so $S(t)f=(\Pi_0+\Pi_1+\Pi_{-1}+\Pi_E+\Pi_M)\,S(t)f=\Pi_E S(t)f$, and hence $\Pi_E\mathscr{H}$ reduces $S(t)$. That the other subspaces of (4.67) reduce $S(t)$ is demonstrated in the same way as are the corresponding results for $S^*(t)$.

The solution $f(x,t)=S(t)\,f(x)$ of problem (0.2)–(0.4) for general initial data $f\in\mathscr{H}$ thus consists of the five parts shown in (4.68). The first term $\Pi_0 f$ is the static part of the solution, the second two terms $S(t)\,\Pi_1 f+S(t)\,\Pi_{-1} f$ are superpositions of reflected plane-wave modes, the fourth term $S(t)\,\Pi_E f$ is a transverse-electric surface wave, i.e., $[S(t)\,\Pi_E f\,]_3$, the normal component of the electric field, is zero, and, finally, the fifth term $S(t)\,\Pi_M f$ constitutes a transverse-magnetic surface wave, i.e., $[S(t)\,\Pi_M f\,]_6$, the normal component of the magnetic field, is zero.

Finally, we observe that for the functions $Z_1(\cdot\,;\mu)$, $Z_2(\cdot\,;\nu)$ of $(2.18_{\mu,\nu})$ the waves $S(t)\,Z_1(\cdot\,;\mu)$, $S^*(t)\,Z_1(\cdot\,;\mu)$ have no E component, i.e., $0=\Pi_E Z_1(\cdot\,;\mu)=\Pi^*_E Z_1(\cdot\,;\mu)$, while the waves $S(t)\,Z_2(\cdot\,;\nu)$, $S^*(t)\,Z_2(;\nu)$ have no M component. Because of (2.20), however, neither $S(t)\,Z_1(\cdot\,;\mu)$ nor $S(t)\,Z_2(\cdot\,;\nu)$ is a solution of Maxwell's equations in $\mathscr{H}$.

5. The Structure of the Surface Waves

In this section we describe the structure of the surface waves and show how data in the subspaces $\Pi_E\mathscr{H}$, $\Pi_M\mathscr{H}$ giving rise to pure surface waves can be constructed.

With $\Delta_2=\partial_1^2+\partial_2^2$ we define the differential operators

$$\begin{aligned}
\mathscr{E}(D)&={}^{t}(-\alpha\mu^{-1}D_2D_3,\,\alpha\mu^{-1}D_1D_3,\,0,\,\mu^{-1}D_1D_3,\,\mu^{-1}D_2D_3,\,\mu^{-1}\Delta_2),\\
\mathscr{E}'(D)&={}^{t}(\bar\alpha\mu^{-1}D_2D_3,\,-\bar\alpha\mu^{-1}D_1D_3,0,\,\mu^{-1}D_1D_3,\,\mu^{-1}D_2D_3,\,\mu^{-1}\Delta_2),\qquad(5.1)\\
\mathscr{M}(D)&={}^{t}(-\varepsilon^{-1}D_1D_3,-\varepsilon^{-1}D_2D_3,-\varepsilon^{-1}\Delta_2,-\alpha^{-1}\varepsilon^{-1}D_2D_3,\alpha^{-1}\varepsilon^{-1}D_1D_3,0),\\
\mathscr{M}'(D)&={}^{t}(\varepsilon^{-1}D_1D_3,\,\varepsilon^{-1}D_2D_3,\,\varepsilon^{-1}\Delta_2,-\varepsilon^{-1}\bar\alpha^{-1}D_2D_3,\,\varepsilon^{-1}\bar\alpha^{-1}D_1D_3,\,0).
\end{aligned}$$

For $f \in \mathscr{H}$ we further define

$$l_E(\xi; f) = {}^{\mathrm{t}}\mathscr{E}'(\xi) E \int_{-a}^{\infty} \exp[i\tau_e(y_3 + a)]\, \Phi_2 f(\xi, y_3)\, dy_3,$$

$$l'_E(\xi; f) = {}^{\mathrm{t}}\bar{\mathscr{E}}(\xi) E \int_{-a}^{\infty} \exp[-i\bar{\tau}_e(y_3 + a)]\, \Phi_2 f(\xi, y_3)\, dy_3,$$

$$l_M(\xi; f) = {}^{\mathrm{t}}\mathscr{M}'(\xi) E \int_{-a}^{\infty} \exp[i\tau_m(y_3 + a)]\, \Phi_2 f(\xi, y_3)\, dy_3,$$

$$l'_M(\xi; f) = {}^{\mathrm{t}}\bar{\mathscr{M}}(\xi) E \int_{-a}^{\infty} \exp[-i\bar{\tau}_m(y_3 + a)]\, \Phi_2 f(\xi, y_3)\, dy_3, \tag{5.2}$$

$$\Phi_2 L_E(\xi, x_3, t; f) = i\mu q(c\alpha\varepsilon)^{-1} |\xi|^{-3} \exp[i\tau_e(x_3 + a) - ite(\xi)]\, l_E(\xi; f),$$

$$\Phi_2 L'_E(\xi, x_3 t; f) = -i\bar{q}\mu(c\bar{\alpha}\varepsilon)^{-1} |\xi|^{-3} \exp[-i\bar{\tau}_e(x_3 + a) + it\bar{e}(\xi)]\, l'_E(\xi; f),$$

$$\Phi_2 L_M(\xi, x_3, t; f) = i\varepsilon p\alpha\, |\xi|^{-3} \exp[i\tau_m(x_3 + a) - itm(\xi)]\, l_M(\xi; f),$$

$$\Phi_2 L'_M(\xi, x_3 t; f) = -i\varepsilon\bar{p}\bar{\alpha}\, |\xi|^{-3} \exp[-i\bar{\tau}_m(x_3 + a) + it\bar{m}(\xi)]\, l'_M(\xi; f).$$

THEOREM 5.1. *In terms of the scalar functions* (5.2) *the E and M surface waves for Maxwell's equations $S(t)\Pi_E f$, $S(t)\Pi_M f$, and their adjoints $S^*(t)\Pi_E^* f$, $S^*(t)\Pi_M^* f$ can be represented in the form*

$$S(t)\Pi_E f(x) = \mathscr{E}(D)\, L_E(x, t; f),$$

$$S(t)\Pi_M f(x) = \mathscr{M}(D)\, L_M(x, t; f),$$

$$S^*(t)\Pi_E^* f(x) = \mathscr{E}'(D)\, L'_E(x, t; f),$$

$$S^*(t)\Pi_M^* F(x) = \mathscr{M}'(D)\, L'_M(x, t; f).$$

The functions $L_E(x, t; f)$, $L_M(x, t; f)$, $L'_E(x, t; f)$, and $L'_M(x, t; f)$ all satisfy the scalar wave equation $(\partial_t^2 - c^2\Delta)\, u(x, t) = 0$ in R^3_{-a} and the respective dissipative-boundary conditions on $\{x_3 = -a\}$

$$\alpha\partial_3 L_E(x', -a, t; f) - \mu\partial_t L_E(x', -a, t; f) = 0,$$

$$\partial_3 L_M(x', -a, t; f) - \alpha\varepsilon\partial_t L_M(x', -a, t; f) = 0, \tag{5.4}$$

$$\bar{\alpha}\partial_3 L'_E(x', -a, t; f) - \mu\partial_t L'_E(x', -a, t; f) = 0,$$

$$\partial_3 L'_M(x', -a, t; f) - \bar{\alpha}\varepsilon\partial_t L'_M(x', -a, t; f) = 0.$$

Conversely, solutions of $(\partial_t^2 - c^2\Delta)\, u(x, t) = 0$ in R^3_{-a} satisfying the respective-boundary conditions (5.4) *and with corresponding initial data $L_E(x, 0; f)$, $\partial_t L_E(x, 0; f)$,... for $f \in \mathscr{D}(R^3_{-a}, C^6)$, say, give rise to surface waves* (5.3) *in the obvious way. Further, if $l(\xi)$ is any measurable function* *from R^2 to C such that $|\xi|^{3/2}\, l(\xi) \in L_2(R^2, C)$ and the functions $L_E(x)$, $L_M(x)$, $L'_E(x)$, and $L'_M(x)$ are defined by setting $t=0$ in*

$$\begin{aligned}
\Phi_2 L_E(\xi, x_3, t) &= \exp[i\tau_e(\xi)(x_3+a) - ite(\xi)]\, l(\xi),\\
\Phi_2 L_M(\xi, x_3, t) &= \exp[i\tau_m(\xi)(x_3+a) - itm(\xi)]\, l(\xi),\\
\Phi_2 L'_E(\xi, x_3, t) &= \exp[-i\bar{\tau}_e(\xi)(x_3+a) + it\bar{e}(\xi)]\, l(\xi),\\
\Phi_2 L'_M(\xi, x_3, t) &= \exp[-i\bar{\tau}_m(\xi)(x_3+a) + it\bar{m}(\xi)]\, l(\xi),
\end{aligned} \tag{5.5}$$

then $\mathscr{E}(D)\, L_E(x)$, $\mathscr{M}(D)\, L_M(x)$, $\mathscr{E}'(D)\, L'_E(x)$, and $\mathscr{M}'(D)\, L'_M(x)$ are contained in $\Pi_E\mathscr{H}$, $\Pi_M\mathscr{H}$, $\Pi_E^*\mathscr{H}$, and $\Pi_M^*\mathscr{H}$, respectively. The corresponding surface waves are

$$\begin{aligned}
[S(t)\, \mathscr{E}(D)\, L_E](x) &= \mathscr{E}(D)\, L_E(x, t),\\
[S(t)\, M(D)\, L_M](x) &= \mathscr{M}(D)\, L_M(x, t),\\
[S^*(t)\, \mathscr{E}'(D)\, L'_E](x) &= \mathscr{E}'(D)\, L'_E(x, t),\\
[S^*(t)\, M'(D)\, L'_M](x) &= \mathscr{M}'(D)\, L'_M(x, t).
\end{aligned} \tag{5.6}$$

Of course, the functions $L_E(x, t)$,... of (5.5) *are solutions of the scalar-wave equation in R^3_{-a} satisfying the respective boundary conditions* (5.4).

Proof. Representations (5.3) can be verified directly from (4.58). That the functions $L_E(x, t; f)$,... satisfy the scalar-wave equation in R^3_{-a} follows by direct computation and the definition of τ_e, τ_m. The fact that these functions satisfy boundary conditions (5.4) follows from Lemma 3.1, while the converse assertion follows immediately from the fact that boundary conditions (5.4) are dissipative for the wave equation: if we form the quadratic forms $F_E(t)$, $F'_E(t)$, $F_M(t)$, and $F'_M(t)$ by affixing the corresponding subscript and eventual prime to the form

$$F(t) = \int_{R^3_{-a}} [|f_t|^2 + c^2\, |\nabla f|^2]\, dx,$$

then for solutions f of the wave equation $\partial_t^2 f - c^2 \Delta f = 0$ in R^3_{-a}

$$\partial_t F_E(t) = \partial_t F'_E(t) = -\mu^{-1}(\alpha + \bar{\alpha}) \int_{R^2} |\partial_3 f(x', -a)|^2\, dx' \leqslant 0,$$

$$\partial_t F_M(t) = \partial_t F'_M(t) = -\varepsilon(\alpha + \bar{\alpha}) \int_{R^2} |\partial_t f(x', -a)|^2\, dx' \quad \leqslant 0.$$

To check that, e.g., $\mathscr{M}'(D) L'_M(x) \in \Pi^*_M \mathscr{H}$ we observe that by (4.47) and (5.5)

$$\Phi_2 M'(D) L'_M(\xi, y_3) = \bar{M}'(\xi) \exp[-i\bar{\tau}_m(y_3 + a)]\, l(\xi),$$

and by (4.48) and (3.23)

$${}^t\mathscr{M}(\xi) E \bar{\mathscr{M}}'(\xi)\, l(\xi) \int_{-a}^{\infty} \exp[-2i\bar{\tau}_m(y_3 + a)]\, dy_3 = i\varepsilon^{-1}\bar{\alpha}^{-1}\bar{p}^{-1} |\xi|^3\, l(\xi).$$

Hence, by (4.64)

$$\Phi_2 \Pi^*_M \mathscr{M}'(D) L'_M(\xi, x_3) = \Phi_2 M'(D) L'_M(\xi, x_3),$$

i.e., $\mathscr{M}'(D) L'_M(x) \in \Pi^*_M \mathscr{H}$. Note that in the case $L'_M(x) = L'_M(x; f)$ this gives a direct proof that Π^*_M is a projection. It is a direct computation to show that $\mathscr{M}'(D) L'_M(x)$ is square integrable over R^3_{-a} if $|\xi|^{3/2}\, l(\xi) \in L_2(R^2, C)$. Assertions (5.6) are obvious.

6. Concluding Remarks

Although in the selfadjoint plane-boundary problems previously considered [5, 12–14] the spectral theorem simplifies certain tasks in obtaining a representation theorem such as Theorem 4.16, it is really not needed: we have not shown that Λ above is "spectral" or even that the set $\sigma(\Lambda)$ of (3.24) is, in fact, the spectrum of Λ. These questions are now academic (in both senses). It may nevertheless be instructive (as part of our general self-improvement program) to investigate which concepts of abstract spectral theory find application here.

It was seen in Section 2 that Λ is dissipative provided only that $\operatorname{Re} \alpha \geqslant 0$. The case $\alpha \equiv 0$ is the selfadjoint case of the classical boundary condition considered, in particular, in [12], while the case $\alpha = i\alpha_2$, $0 \neq \alpha_2 \in R$, is the selfadjoint case considered in [5]. The only situation so far not covered is, thus, $\alpha_1 = \operatorname{Re} \alpha > 0$, $\alpha_2 = \operatorname{Im} \alpha = 0$. It appears that in this case some very peculiar things may happen depending on the values of the parameters α_1, ε, μ. This case will be considered in a separate report.

In the classical case ($\alpha \equiv 0$) there are no surface waves [12]. If, however, the layer $R^2 \times (-a, 0)$ in the medium of Section 0 is replaced by a uniform dielectric with parameters ε_1, μ_1 such that $(\varepsilon_1\mu_1)(\varepsilon\mu)^{-1} > 1$, then there arise two infinite sequences of surface waves which propagate along the interface $\{x_3 = 0\}$ [14]. When the boundary condition with $\alpha \equiv 0$ is replaced by that of Lemma 3.1 it is an interesting question whether both types of surface waves persist—those propagating along $\{x_3 = 0\}$ and those propagating along $\{x_3 = -a\}$.

Finally, the Cauchy problem in R^3 with R^3_{-a} as above and the half space $\{x_3 < -a\}$ consisting of a medium with distinct electromagnetic parameters and finite conductivity is likewise solved by a contractive semigroup $\tilde{S}(t)$ [6]. As mentioned in Section 0, it is the intention of the boundary condition (0.4) that $S(t)$ of Theorem 4.13 approximate $\tilde{S}(t)$ in R^3_{-a} in some sense, at least for some class of initial data in $\mathscr{H}$. Our results already indicate that this approximation is much less felicitous than heretofore believed. The question can be vaguely formulated, e.g., as follows: consider functions in $\mathscr{H}$ extended to R^3 by zero, and let $C \subset \mathscr{H}$ be some family of functions with compact support in R^3_{-a}. For what C, if any, is it true that for $\phi \in C$ there exists $f \in \mathscr{H}$ such that $\tilde{S}(t)\phi \sim S(t)f$ in some sense (on compact sets for large times, say)? This should be a very interesting and challenging problem.

Appendix A

We shall show here that the operator Π_0 of (4.36), defined initially on functions $\mathscr{H} \ni f = \chi_a F$ which are the restrictions to R^3_{-a} of functions $F \in \mathscr{H}$ such that $\hat{F} \in \mathscr{D}(R^3, C^6)$, $\{|\xi| = 0\} \cap \operatorname{supp} \hat{F} = \varnothing$, extends to a bounded, selfadjoint projection on $\mathscr{H}$. We use the notation $f_a(x) = f(x', x_3 - a)$, $x \in R^3_+$, and, noting that $\exp(-ia\rho)\,\hat{F}(\eta) = \hat{F}_a(\eta)$, we write (4.36) for $x \in R^3_+$ in the form

$$(\Pi_0 f)_a\,(x) = \Phi_3^* P_0 \hat{F}_a(x) - (2\pi)^{-3/2} \int \exp(ix'\xi - |\xi|\, x_3)\, P_0^r(\omega)\, \hat{F}_a(\eta)\, d\eta. \tag{A.1}$$

We define

$$\tilde{F}_\pm(\xi, \rho) = \int_{R_\pm} \exp(-i\rho x_3)\, \Phi_2 F_a(\xi, x_3)\, dx_3 \tag{A.2}$$

and note that for each ξ $\tilde{F}_\pm(\xi, \rho) \in H_2^\mp(R, C^6)$, the Hardy spaces of functions regular in the lower $(-)$ and upper $(+)$ half planes. Equation (A.1) can now be written

$$\begin{aligned}\Phi_2(\Pi_0 f)_a\,(\xi, x_3) = {} & (2\pi)^{-1} \int \exp(i\rho x_3)\, P_0(\xi, \rho)[\tilde{F}_+(\xi, \rho) + \tilde{F}_-(\xi, \rho)]\, d\rho \\ & - (2\pi)^{-1} \int \exp(-|\xi|\, x_3)\, P_0^r(\xi, \rho)[\tilde{F}_+(\xi, \rho) + \tilde{F}_-(\xi, \rho)]\, d\rho,\end{aligned}$$

$$P_0(\xi, \rho) = (\rho^2 + |\xi|^2)^{-1}\, \tilde{P}_0(\xi, \rho), \qquad P_0^r(\xi, \rho) = (\rho^2 + |\xi|^2)^{-1}\, \tilde{P}_0^r(\xi, \rho),$$

$$\tilde{P}_0(\xi, \rho) = \begin{pmatrix} {}^t\eta\eta & 0 \\ 0 & {}^t\eta\eta \end{pmatrix}, \qquad \tilde{P}_0^r(\xi, \rho) = \begin{pmatrix} d(\xi)\eta & 0 \\ 0 & d(\xi)\eta \end{pmatrix},$$

$$d(\xi) = {}^t(\xi_1, \xi_2, i\,|\xi|), \qquad \eta = (\xi, \rho). \tag{A.3}$$

We observe that F_a is smooth, rapidly decreasing, and, on integrating by parts, for each fixed $\xi \in R^2$

$$|\rho \tilde{F}_{\pm}(\xi, \rho)| \leqslant |\Phi_2 F_a(\xi, 0)| + |D_3 \Phi_2 F_a(\xi, \cdot)|_{L_1}. \tag{A.4}$$

By the residue theorem for each fixed ξ

$$\begin{aligned} \lim_{N \uparrow \infty} \Bigg\{ \int_{-N}^{N} & \exp(i\rho x_3)\, P_0(\xi, \rho)\, \tilde{F}_-(\xi, \rho)\, d\rho \\ & + \int_{\{|\zeta| = N\} \cap R_+^2} \exp(i\zeta x_3)\, P_0(\xi, \rho)\, \tilde{F}_-(\xi, \zeta)\, d\zeta \Bigg\} \\ & = \pi\, |\xi|^{-1} \exp(-|\xi|\, x_3)\, \tilde{P}_0(\xi, i\,|\xi|)\, \tilde{F}_-(\xi, i\,|\xi|) \end{aligned}$$

and by (A.4)

$$\begin{aligned} \left| \int_{\{|\zeta| = N\} \cap R_+^2} \exp(i\zeta x_3)\, P_0(\xi, \zeta)\, \tilde{F}_-(\xi, \zeta)\, d\zeta \right| \\ \leqslant \text{const} \int_0^{\pi} \exp(-N x_3 \sin \phi)\, d\phi = o_N(1). \end{aligned}$$

Hence,

$$\int \exp(i\rho x_3)\, P_0(\xi, \rho)\, \tilde{F}_-(\xi, \rho)\, d\rho = \pi\, |\xi|^{-1} \exp(-|\xi|\, x_3)\, \tilde{P}_0(\xi, i\,|\xi|)\, \tilde{F}_-(\xi, i\,|\xi|), \tag{A.5}$$

and, similarly,

$$\begin{aligned} \int & \exp(-|\xi|\, x_3)\, P_0^r(\xi, \rho)\, \tilde{F}_-(\xi, \rho)\, d\rho \\ & = \pi\, |\xi|^{-1} \exp(-|\xi|\, x_3)\, \tilde{P}_0(\xi, i\,|\xi|)\, \tilde{F}_-(\xi, i\,|\xi|). \end{aligned} \tag{A.6}$$

Thus, from (A.3), (A.5), and (A.6)

$$\begin{aligned} \Phi_2(\Pi_0 f)_a\, (\xi, x_3) = (2\pi)^{-1} & \int \exp(i\rho x_3)\, P_0(\xi, \rho)\, \tilde{F}_+(\xi, \rho)\, d\rho \\ & - (2\pi)^{-1} \int \exp(-|\xi|\, x_3)\, P_0^r(\xi, \rho)\, \tilde{F}_+(\xi, \rho)\, d\rho. \end{aligned} \tag{A.7}$$

Again by the residue theorem

$$\int P_0^r(\xi, \rho)\, \tilde{F}_+(\xi, \rho)\, d\rho = \pi\, |\xi|^{-1}\, \tilde{P}_0^r\, (\xi, -i\,|\xi|)\, \tilde{F}_+(\xi, -i\,|\xi|), \tag{A.8}$$

and hence

$$\Phi_2(\Pi_0 f)_a(\xi, x_3) = (2\pi)^{-1} \int \exp(i\rho x_3)\, P_0(\xi, \rho)\, \tilde{F}_+(\xi, \rho)\, d\rho$$
$$- 2^{-1} |\xi|^{-1} \exp(-|\xi|\, x_3)\, \tilde{P}_0^r(\xi, -i|\xi|)\, \tilde{F}_+(\xi, -i|\xi|), \quad \text{(A.9)}$$
$$(\Pi_0 f)_a(x) = (2\pi)^{-1/2}\, \Phi_3^* P_0 \tilde{F}_+(x) - (2\pi)^{-1}\, 2^{-1} \int_{R^2} \exp(ix'\xi - |\xi|\, x_3)$$
$$\times |\xi|^{-1}\, \tilde{P}_0^r(\xi, -i|\xi|)\, \tilde{F}_+(\xi, -i|\xi|)\, d\xi.$$

Now from (A.2), (A.9)

$$2|\Pi_0 f|^2_{\mathscr{H}} \leqslant (2\pi)^{-1} |\chi_+ F_a|^2_{\mathscr{H}}$$
$$+ 4^{-1} \int_{R^2} d\xi \int_0^\infty dx_3\, |\xi|^{-2}$$
$$\times \exp(-2|\xi|\, x_3)\, |E\tilde{P}_0^r(\xi, -i|\xi|)\, \tilde{F}_+(\xi, -i|\xi|)|^2$$
$$\leqslant (2\pi)^{-1} |f|^2_{\mathscr{H}} + 8^{-1} \int_{R^2} d\xi\, |\xi| \left[\int_0^\infty \exp(-2|\xi|\, y_3)\, dy_3\right]$$
$$\times \left[\int_0^\infty |E\Phi_2 F_a(\xi, y_3)|^2\, dy_3\right]$$
$$< (2\pi)^{-1} |f|^2_{\mathscr{H}} + 8^{-1} \int_{R^2} d\xi \int_0^\infty |E\Phi_2 F_a(\xi, y_3)|^2\, dy_3$$
$$= (2\pi)^{-1} |f|^2_{\mathscr{H}} + 8^{-1} |f|^2_{\mathscr{H}},$$

and, hence, Π_0 is bounded on a dense set of of $f \in \mathscr{H}$.

We now note that ${}^{\mathrm{t}}[E\tilde{P}_0^r(\xi, -i|\xi|)] = E\tilde{P}_0^r(\xi, -i|\xi|)$; by (A.9)

$$(f, \Pi_0 g) = (\chi_+ f_a, (\Pi_0 g)_a)_{\mathscr{H}}$$
$$= (2\pi)^{-1/2} (\chi_+ f_a, \Phi_3^* P_0 \tilde{G}_+)_{\mathscr{H}}$$
$$-2^{-1}(\Phi_2 \chi_+ f_a, |\cdot|^{-1} \exp(-|\cdot|\, \cdot)\, \tilde{P}_0^r(\cdot, -i|\cdot|))\, \tilde{G}_+(\cdot, -i|\cdot|))_{\mathscr{H}}$$
$$= (2\pi)^{-1/2} (\Phi_3^* P_0 \tilde{F}_+, \chi_+ g_a)_{\mathscr{H}}$$
$$- 2^{-1} \int_{R^2}^\infty \int_0^\infty {}^{\mathrm{t}}[|\xi|^{-1}\, \tilde{P}_0^r(\xi, -i|\xi|)\, \tilde{F}_+(\xi, -i|\xi|)]E$$
$$\times \exp(-|\xi|\, y_3)\, \Phi_2\, g(\xi, y_3)\, dy_3\, d\xi$$
$$= (\Pi_0 f, g),$$

and hence Π_0 is selfadjoint in $\mathscr{H}$.

To verify Π_0 is a projection, we write (A.9) as

$$\begin{aligned}\Phi_2(\Pi_0 f)_a(\xi, x_3) = \chi_+(x_3)\{&\Phi_1^* P_0(\xi, \cdot)\, \Phi_1 \chi_+ \Phi_2 f_a(\xi, \cdot)(x_3) \\ &- 2^{-1} |\xi|^{-1} \exp(-|\xi|\, x_3)\, \tilde{P}_0^r(\xi, -i\,|\xi|)(2\pi)^{1/2} \\ &\times \Phi_1 \chi_+ \Phi_2 f_a(\xi, -i\,|\xi|). \qquad \text{(A.10)}\end{aligned}$$

We must check that

$$\begin{aligned}\Phi_2(\Pi_0^2 f)_a(\xi, x_3) = \chi_+(x_3)\{&\Phi_1^* P_0(\xi, \cdot)\, \Phi_1 \chi_+ \Phi_2(\Pi_0 f)_a(\xi, \cdot)(x_3) \\ &- 2^{-1} |\xi|^{-1} \exp(-|\xi|\, x_3)\, \tilde{P}_0^r(\xi, -i\,|\xi|)(2\pi)^{1/2} \qquad \text{(A.11)} \\ &\times \Phi_1 \chi_+ \Phi_2(\Pi_0 f)_a(\xi, -i\,|\xi|)\end{aligned}$$

is equal to $\Phi_2(\Pi_0 f)_a$. This is a straightforward but lengthy computation. We substitute (A.10) into (A.11) and denote by (i, j), $i, j = 1, 2$, the resulting term of the expansion involving the ith term of (A.11) and jth term of (A.10). We shall compute the term (1, 1) in detail and merely state the results for (1, 2), (2, 1), and (2, 2). Thus,

$$\begin{aligned}(1, 1) &= \chi_+(x_3)\, \Phi_1^* P_0(\xi, \cdot)\, \Phi_1 \chi_+ \Phi_1^* P_0(\xi, \cdot)\, \Phi_1 \chi_+ \Phi_2 f_a(\xi, \cdot)(x_3) \\ &= \chi_+(x_3)\, \Phi_1^* P_0(\xi, \cdot)\, \Phi_1 \chi_+ \Phi_2 f_a(\xi, \cdot)(x_3) \qquad \text{(A.12)} \\ &\quad - \chi_+(x_3)\, \Phi_1^* P_0(\xi, \cdot)\, \Phi_1 \chi_- \Phi_1^* P_0(\xi, \cdot)\, \Phi_1 \chi_+ \Phi_2 f_a(\xi, \cdot)(x_3).\end{aligned}$$

Setting $\tilde{f}_a^+(\xi, \sigma) = \Phi_1 \chi_+ \Phi_2 f_a(\xi, \sigma)$ ($\in H_2^-$ with respect to σ for fixed ξ) and using the residue theorem, we compute the second term of (A.12)

$$\begin{aligned}&\chi_-(y_3)\, \Phi_1^* P_0(\xi, \cdot)\, \tilde{f}_a^+(\xi, y_3) \\ &\quad = \chi_-(y_3)(2\pi)^{-1/2} \int e^{i\sigma y_3} P_0(\xi, \sigma)\, \tilde{f}_a^+(\xi, \sigma)\, d\sigma \\ &\quad = \chi_-(y_3)(2\pi)^{1/2}\, |\xi|^{-1} \exp(|\xi|\, y_3)\, 2^{-1} \tilde{P}_0(\xi, -i\,|\xi|)\, \tilde{f}_a^+(\xi, -i\,|\xi|),\end{aligned}$$

and hence

$$\begin{aligned}&\Phi_1 \chi_- \Phi_1^* P_0(\xi, \cdot)\, \tilde{f}_a^+(\xi, \cdot)(\rho) \\ &\quad = i\,|\xi|^{-1}\, 2^{-1}(\rho + i\,|\xi|)^{-1}\, \tilde{P}_0(\xi, -i\,|\xi|)\, \tilde{f}_a^+(\xi, -i\,|\xi|);\end{aligned}$$

now

$$\begin{aligned}&P_0(\xi, \rho)\, \Phi_1 \chi_- \Phi_1^* P_0(\xi, \cdot)\, \tilde{f}_a^+(\xi, \cdot)(\rho) \\ &\quad = 2^{-1} |\eta|^{-2} \begin{pmatrix} {}^t\eta(\xi, -i\,|\xi|) & 0 \\ 0 & {}^t\eta(\xi, -i\,|\xi|) \end{pmatrix} \tilde{f}_a^+(\xi, -i\,|\xi|),\end{aligned}$$

and hence

$$-\chi_+(x_3)\,\Phi_1^* P_0(\xi,\cdot)\,\Phi_1 \chi_- \tilde{P}_0(\xi,\cdot)\,\tilde{f}_a^+(x_3)$$
$$= -4^{-1}\chi_+(x_3)\,|\xi|^{-1}\exp(-|\xi|\,x_3)\,\tilde{P}_0^r(\xi,-i\,|\xi|)(2\pi)^{1/2}\,\tilde{f}_a^+(\xi,-i\,|\xi|),$$

so, finally,

$$(1,1) = \chi_+(x_3)\,\Phi_1^* P_0(\xi,\cdot)\,\tilde{f}_a^+(\xi,\cdot)(x_3) - 4^{-1}\,|\xi|^{-1}\,\chi_+(x_3)\exp(-|\xi|\,x_3)\,\tilde{P}_0^r(\xi,-i\,|\xi|)(2\pi)^{1/2}\,\tilde{f}_a^+(\xi,-i\,|\xi|). \tag{A.13}$$

In a similar way,

$$\begin{aligned}(1,2) &= -4^{-1}\chi_+(x_3)\,|\xi|^{-1}\exp(-|\xi|\,x_3)\,\tilde{P}_0^r(\xi,-i\,|\xi|)(2\pi)^{1/2}\,\tilde{f}_a^+(\xi,-i\,|\xi|),\\ (2,1) &= (1,2),\\ (2,2) &= -(1,2).\end{aligned} \tag{A.14}$$

Hence, adding (A.13) and (A.14), from (A.11) we have $\Phi_2(\Pi_0^2 f)_a(\xi, x_3) = \Phi_2(\Pi_0 f)_a(\xi, x_3)$, i.e., $\Pi_0^2 = \Pi_0$.

Appendix B

We shall now prove Lemma 4.10. The proof consists of two parts: the first part is to show that the limit $\kappa \downarrow 0$ can be taken under the integrals in (4.35); the second part is to evaluate the limits using the well-known formula

$$\lim_{\kappa\downarrow 0}\int_a^b \frac{\varepsilon}{(\lambda-\mu)^2+\kappa^2}\,\phi(\lambda)\,d\lambda = \pi\chi_{(a,b)}(\mu)\,\phi(\mu) \tag{B.1}$$

for any continuous function $\phi(\mu)$. We extend g to R^3 by zero, and we choose $N > N_0$ such that

$$S \equiv R_\xi^2 \cap \operatorname{supp}\hat{F} \subset B_N = \{|\xi| < N\}. \tag{B.2}$$

From (4.32)–(4.34) we have

$$\begin{aligned}H(\eta;\zeta) &\equiv {}^t[\Psi(\zeta)\,g(\eta)]\,E\Psi'(\zeta)\,f(\eta)\\ &= \sum_{j=-1}^{1} {}^t[\Phi g_a(\eta)]\,EP_j\Phi\chi_+F_a(\eta)\,|\zeta-\lambda_j(\eta)|^{-2}\\ &\quad+ \sum_{j=-1}^{1} {}^t[\Phi g_a(\eta)]\,EP_jE^{-1}\,{}^t[EM'(\eta;\zeta)]\,\tilde{f}_a(\xi;\bar{\zeta})[\zeta-\lambda_j(\eta)]^{-1}\\ &\quad+ \sum_{j=-1}^{1} {}^t\tilde{g}_a(\xi;\bar{\zeta})\,EM(\eta;\zeta)\,\Phi\chi_+F_a(\eta)[\bar{\zeta}-\lambda_j(\eta)]^{-1}\\ &\quad+ {}^t\tilde{g}_a(\xi;\bar{\zeta})\,EM(\eta;\zeta)\,E^{-1}\,{}^t[EM'(\eta;\zeta)]\,\tilde{f}_a(\xi;\bar{\zeta}) \equiv \sum_{i=1}^{4} H_i(\eta;\zeta).\end{aligned} \tag{B.3}$$

We observe that both $\tilde{g}_a(\xi;\bar{\zeta})$ and $\tilde{f}_a(\xi;\bar{\zeta})$ are bounded uniformly with respect to ζ in the cut plane ($\operatorname{Im}\tau \geqslant 0$), and the support of $\tilde{f}_a(\xi;\bar{\zeta})$ in ξ is compact and equal to S of (B.2).

Part 1. *Interchange of the Integration Over* R^3 *and* $\lim_{\kappa\downarrow 0}$

We consider the first term of (4.35),

$$\begin{aligned} I(\kappa) &= \int_{R^3} d\eta\, h(\eta,\kappa;\delta,N), \\ h(\eta,\kappa;\delta,N) &= \int_\delta^N \kappa H(\eta;\lambda \pm i\kappa)\, d\lambda \\ &= \sum_{i=1}^4 \int_\delta^N \kappa H_i(\eta;\lambda \pm i\kappa)\, d\lambda \\ &\equiv \sum_{i=1}^4 h_i(\eta,\kappa;\delta,N), \end{aligned} \tag{B.4}$$

and we write

$$\begin{aligned} I(\kappa) &= \int_{\{c|\eta|>2N\}} d\eta\, h(\eta,\kappa;\delta,N) + \int_{\{c|\eta|<2N\}} d\eta\, h(\eta,\kappa;\delta,N) \\ &\equiv I_>(\kappa) + I_<(\kappa). \end{aligned} \tag{B.5}$$

It will be shown that the integrands of $I_>$ and $I_<$ are bounded by integrable functions uniformly with respect to $\kappa \in (0,\kappa_0]$. Below C is a generic constant depending only on δ and N. We first consider $I_>(\kappa)$. In $H_1(\eta;\kappa)$ $|\zeta - \lambda_j(\eta)|$ is bounded away from zero for $\lambda \in [\delta, N]$, $c|\eta| > 2N$, so

$$h_1(\eta,\kappa;\delta,N) \leqslant \kappa C\, |\Phi g_a(\eta)|\, |\Phi \chi_+ F_a(\eta)|. \tag{B.6}$$

In $H_i(\eta;\kappa)$, $i = 2, 3, 4$,

$$\begin{aligned} |\rho \pm \tau(\xi,\zeta)| &\geqslant ||\rho| - |\tau(\xi,\zeta)|| \\ &= ||\rho| - c^{-1}[(\lambda^2 - c^2|\xi|^2)^2 + 2(\lambda^2 + c^2|\xi|^2)\kappa^2 + \kappa^4]^{1/4}| \\ &\equiv l(\xi,\rho;\zeta). \end{aligned}$$

Now $||\rho| - |\tau(\xi,\zeta)||$ vanishes only for $\zeta = c|\eta| \pm i0$, so for $c|\eta| > 2N$, $\lambda \in [\delta, N]$, $|\xi| \leqslant N$, $\kappa \in [0,\kappa_0]$, it is bounded away from zero for $|\rho| \leqslant R$, and for R sufficiently large and $|\rho| > R$ it is $O(|\rho|)$ uniformly with respect to $\kappa \in (0,\kappa_0]$. Further,

$$|\tau(\xi,\zeta)| \geqslant C\,|\zeta - c|\xi||^{1/2} \geqslant C\kappa^{1/2} \tag{B.8}$$

for $\lambda \in [\delta, N]$ and all $|\xi|$. Therefore,

$$
\begin{aligned}
|h_2(\eta, \kappa; \delta, N)| &\leqslant C\,|\Phi g_a(\eta)|, \\
|h_3(\eta, \kappa; \delta, N)| &\leqslant C\,|\Phi \chi_+ F_a(\eta)|, \\
\sup_{\xi \in S} |h_4(\eta, \kappa; \delta, N)| &\leqslant C(1+\rho^2)^{-1}.
\end{aligned}
\tag{B.9}
$$

It follows from (B.7) and (B.9) that the $\lim_{\kappa \downarrow 0}$ may be taken under the integral in $I_>(\kappa)$.

We write

$$I_<(\kappa) = \sum_{i=1}^{4} I_<^i(\kappa) = \sum_{i=1}^{4} \int_{\{c|\eta| < 2N\}} h_i(\eta, \kappa; \delta, N)\, d\eta.$$

Now

$$|h_1(\eta, \kappa; \delta, N)| \leqslant C\left(1 + \int_\delta^N \frac{\kappa}{(\lambda - c|\eta|)^2 + \kappa^2}\, d\lambda\right) \leqslant C(1+\pi),$$

so the limit may be taken under the integral in $I_<^1(\kappa)$. In $I_<^i(\kappa)$, $i = 2, 3, 4$, the only difficulty occurs where $|\eta| \sim |\xi|$, $\kappa \sim 0$, and the singularities $\zeta = c|\eta| \pm i0$ of $[|\rho| - |\tau(\xi, \zeta)|]^{-1}$ and $\zeta = c|\eta| \pm i0$ of $\tau^{-1}(\xi, \zeta)$ approach one another. To deal with this we define

$$
\begin{aligned}
D &= D(\sigma; \delta, N) = \{\eta \in R^3 : c|\eta| \leqslant 2N, 0 < |\rho| \leqslant \sigma, \delta/2 \leqslant |\xi| \leqslant 2N\}, \\
\tilde{D} &= \tilde{D}(\sigma; \delta, N) = \{\eta \in R^3 : c|\eta| \leqslant 2N, |\rho| > \sigma, \delta/2 \leqslant |\xi| \leqslant 2N\},
\end{aligned}
$$

with σ to be specified below. Since $\{|\xi| < \delta/2\} \cap S = \varnothing$ we can write

$$I_<^i(\kappa) = \int_D h_i(\eta, \kappa; \delta, N)\, d\eta + \int_{\tilde{D}} h_i(\eta, \kappa; \delta, N)\, d\eta, \qquad i = 2, 3, 4.$$

There is no problem in taking the limit under the integral over $\tilde{D}$, since the singularities mentioned above are separated. It thus remains to consider

$$J_i(\kappa) = \int_D h_i(\eta, \kappa; \delta, N)\, d\eta. \tag{B.10}$$

For the function $l(\xi, \rho; \zeta)$ of (B.7) we have

$$l(\xi, \rho; c|\eta| \pm i\kappa) = c^{-1}[(c\rho)^4 + 2c^2(\rho^2 + |\xi|^2)\kappa^2 + \kappa^4]^{1/4} - |\rho|$$

which is positive and continuous on $D(\sigma; \delta, N)$, and

$$l(\xi, 0; c|\eta| \pm i\kappa) = c^{-1}\kappa^{1/2}[2c^2|\xi|^2 + \kappa^4]^{1/4} \geqslant C\kappa^{1/2}.$$

Hence, there exists $\sigma_1 > 0$ such that $|\rho| < \sigma_1$ implies

$$l(\xi, \rho; c\,|\eta| \pm i\kappa) \geqslant C\kappa^{1/2}$$

on $D(\sigma_1, \delta; N)$. Hence, on $D(\sigma_1; \delta, N)$ the function

$$h(\eta, \zeta) = [\zeta - \lambda_1(\eta)][\rho \pm \tau(\xi, \zeta)]^{-1} \tag{B.11}$$

satisfies

$$|h(\eta; \lambda_1(\eta) \pm i\kappa)| \leqslant C\kappa^{1/2}.$$

Since $h(\eta, \lambda, \kappa) \equiv h(\eta, \zeta)$ is uniformly continuous on $\bar{D}(\sigma_1, \delta, N) \times [\delta, N]$ for each κ, there thus exists a neighborhood of the diagonal $O_v = \{(c\,|\eta|, \lambda): \lambda \in [c\,|\eta| - v, c\,|\eta| + v]\} \subset \bar{D}(\sigma_1; \delta, N) \times [\delta, N]$ such that

$$|h(\eta, \zeta)| \leqslant C\kappa^{1/2}. \tag{B.12}$$

From (B.8), (B.11), and (B.12) the function

$$\phi(\eta, \lambda, \kappa) \equiv \phi(\eta, \zeta) = h(\eta, \zeta)\, \tau^{-1}(\xi, \zeta)$$

thus satisfies

$$|\phi(\eta, \zeta)| \leqslant C \tag{B.13}$$

on $O_v \times (0, \kappa_0]$. Now choose and fix $0 < \sigma \leqslant \sigma_1$ such that $|\eta| - |\xi| < v/2$. This defines the region of integration $D = D(\sigma; \delta, N)$ in (B.10). Now for $m = 1, 2$ and $\eta \in D(\sigma; \delta, N)$

$$\begin{aligned}
&\int_\delta^N d\lambda \left\{ \frac{\kappa}{|\zeta - c\,|\eta||^2} |\phi(\eta, \zeta)|^m \right\}_{\zeta = \lambda \pm i\kappa} \\
&\quad = \left[\int_\delta^{c|\eta| - v} + \int_{c|\eta| - v}^{c|\eta| + v} + \int_{c|\eta| + v}^{N} \right] d\lambda \left\{ \frac{\kappa}{|\zeta - c\,|\eta||^2} |\phi(\eta, \zeta)|^m \right\}_{\zeta = \lambda \pm i\kappa}.
\end{aligned}$$

The outermost integrals are bounded by constants uniformly on $D(\sigma; \delta, N) \times (0, \kappa_0]$, while by (B.13) for $m = 1, 2$

$$\int_{c|\eta| - v}^{c|\eta| + v} d\lambda \left\{ \frac{\kappa}{|\zeta - c\,|\eta||^2} |\phi(\eta, \zeta)|^m \right\}_{\zeta = \lambda \pm i\kappa} \leqslant C\pi.$$

Now from (B.3), (B.4)

$$|h_i(\eta, \kappa; \delta, N)| \leqslant C \int_\delta^N d\lambda \left\{ \frac{\kappa}{|\zeta - c\,|\eta||^{-2}} |\phi(\eta, \zeta)| \right\}_{\zeta = \lambda \pm i\kappa}, \qquad i = 2, 3,$$

$$|h_4(\eta, \kappa; \delta, N)| \leqslant C \int_\delta^N d\lambda \left\{ \frac{\kappa}{|\zeta - c\,|\eta||^2} |\phi(\eta, \zeta)|^2 \right\}_{\zeta = \lambda \pm i\kappa},$$

and hence $\lim_{\kappa\downarrow 0}$ may be taken under the integral in (B.10). The proof that $\lim_{\kappa\downarrow 0}$ can be taken under the integral in the first term of (4.35) is now complete. The proof for the second terms is exactly the same. The only singularities of the integrands in the last two terms of (4.35) occur at the poles $e(\xi)$ and $m(\xi)$ of $C(\xi,\zeta)$, and to deal with these we write (see Lemma 4.5 and (3.1))

$$\begin{aligned}
&\Sigma_E(\kappa)\\
&\equiv \kappa c q^2 \int_{R^3} d\eta \int_\delta^N {}^t[\Psi(e_\pm(\lambda))\, g(\eta)]\, E\Psi'(\bar{e}_\mp(\lambda))\, f(\eta)\, d\lambda\\
&= \kappa c q^2 \int_{R^3} d\eta \int_\delta^N\\
&\times \frac{{}^t\{[\bar{e}_\pm(\lambda)-\bar{e}(\xi)]\;\Psi(e_\pm(\lambda))\, g(\eta)\}\, E[e_\mp(\lambda)-e(\xi)]\;\Psi'(\bar{e}_\mp(\lambda))\, f(\eta)}{[e_\pm(\lambda)-e(\xi)][e_\mp(\lambda)-e(\xi)]}\, d\lambda\\
&= \int_{R^3} d\eta \int_{c\delta}^{cN} d\lambda\\
&\times \frac{{}^t\{[\bar{e}_\pm(c^{-1}\lambda)-\bar{e}(\xi)]\;\Psi(e_\pm(c^{-1}\lambda))\, g(\eta)\}\, E[e_\mp(c^{-1}\lambda)-e(\xi)]\;\Psi'(\bar{e}_\mp(c^{-1}\lambda))\, f(\eta)}{\kappa^{-1}[(\lambda-c\,|\xi|)^2+\kappa^2]}
\end{aligned}\tag{B.14}$$

$$\begin{aligned}
&\Sigma_M(\kappa)\\
&\equiv \kappa\varepsilon^{-1}p^2 \int_{R^3} d\eta \int_\delta^N {}^t[\Psi(m_\pm(\lambda))\, g(\eta)]\, E\Psi'(\bar{m}_\mp(\lambda))\, f(\eta)\, d\lambda\\
&= \kappa\varepsilon^{-1}p^2 \int_{R^3} d\eta \int_\delta^N\\
&\times \frac{{}^t\{[\bar{m}_\pm(\lambda)-\bar{m}(\xi)]\;\Psi(m_\pm(\lambda))\, g(\eta)\}\, E[m_\mp(\lambda)-m(\xi)]\;\Psi'(\bar{m}_\mp(\lambda))\, f(\eta)}{[m_\pm(\lambda)-m(\xi)][m_\mp(\lambda)-m(\xi)]}\, d\lambda\\
&= \int_{R^3} d\eta \int_{\varepsilon^{-1}\delta}^{\varepsilon^{-1}N} d\lambda\\
&\times \frac{{}^t\{[\bar{m}_\pm(\varepsilon\lambda)-\bar{m}(\xi)]\;\Psi(m_\pm(\varepsilon\lambda))\, g(\eta)\}\, E[m_\mp(\varepsilon\lambda)-m(\xi)]\;\Psi'(\bar{m}_\mp(\lambda))\, f(\eta)}{\kappa^{-1}[(\lambda-\varepsilon^{-1}\,|\xi|)^2+\kappa^2]}.
\end{aligned}$$

The numerators of the integrands here are bounded by integrable functions of η uniformly with respect to $\kappa\in(0,\kappa_0]$, since the only possible difficulty occurs at the poles of $C(\xi,\zeta)$, and here, e.g.,

$$\frac{\Delta_E(\xi, e_\pm(c^{-1}\lambda))}{e_\pm(c^{-1}\lambda)-e(\xi)} = \frac{\mu(q\lambda \pm iq\kappa)+\alpha\tau(\xi, q\lambda\pm iq\kappa)}{q\lambda\pm iq\kappa - cq\,|\xi|} \tag{B.15}$$

is continuous on $S \times [c\delta, cN] \times (0, \kappa_0]$ and at $\lambda = c|\xi|$ as $\kappa \downarrow 0$ it converges to $\varepsilon p^{-2} = -\varepsilon\alpha^2 q^{-2} = \partial_\zeta \Delta_E(\xi, e(\xi))$. The verification of the interchange of limit and integration in this case is thus straightforward. Part 1 of the project is now complete.

Part 2. The Evaluation of $\lim_{\kappa\downarrow 0}$ *in* (4.35)

We first evaluate $\lim_{\kappa\downarrow 0} h(\eta, \kappa)$ of (B.4). In order to use formula (B.1) we must ensure that $H_j(\eta, \lambda \pm i0)$, $j = 2, 3, 4$, are continuous functions in a neighborhood of $c|\eta|$, and we must therefore exclude $c|\xi|$, the zero of $\tau(\xi, \lambda \pm i0)$, from this neighborhood. To this end we define $T(\sigma; \delta, N) = \{\eta \in R^3 : |\rho| \geqslant \sigma,\ c|\eta| \neq \delta, N\}$. Since $h(\eta, \kappa)$ is bounded by an integrable function uniformly with respect $\kappa \in (0, \kappa_0]$,

$$\int_{R^3} h(\eta, \kappa)\, d\eta = \int_{T(\sigma;\delta,N)} h(\eta, \kappa)\, d\eta + o_\sigma(1). \tag{B.16}$$

It is evident from the estimates of Part 1 that $h(\eta, \kappa) = \chi_{(\delta,N)}(c|\eta|)\, h(\eta, \kappa) + o_\kappa(1)$ for $\eta \in T(\sigma; \delta, N)$. Now choose v such that for $|\rho| \geqslant \sigma$, $c|\xi| \notin [c|\eta| - v, c|\eta| + v]$. Then on $T(\sigma; \delta, N)$

$$h(\eta, \kappa) = \chi_{(\delta,N)}(|\eta|) \int_{c|\eta|-v}^{c|\eta|+v} \kappa H(\eta; \lambda \pm i\kappa) + o_\kappa(1).$$

From (B.1), (B.3), (B.4) for $\eta \in T(\sigma; \delta, N)$

$$h_1(\eta, \pm 0) = \pi\chi_{(\delta,N)}(c|\eta|)\, {}^{\mathrm{t}}[\Phi_3\chi_+ g_a(\eta)]\, EP_1(\omega)\, \Phi_3\chi_+ f_a(\eta), \tag{B.17}$$

and

$$\begin{aligned} h_2(\eta, \kappa) &= \chi_{(\delta,N)}(c|\eta|) \sum_{j=-1}^{1} \int_\delta^N \kappa\{[\zeta - \lambda_j(\eta)]^{-1}\, {}^{\bar{\mathrm{t}}}[\Phi_3\chi_+ g_a(\eta)]\, EP_jE^{-1} \\ &\quad \times {}^{\mathrm{t}}EM'(\eta; \zeta)]\, \tilde{f}_a(\xi, \bar{\zeta})\}_{\zeta=\lambda\pm i\kappa}\, d\lambda + o_\kappa(1) \\ &= \chi_{(\delta,N)}(c|\eta|) \int_\delta^N d\lambda \left\{ \frac{\kappa}{|\zeta - \lambda_1(\eta)|^2}\, {}^{\bar{\mathrm{t}}}[\Phi_3\kappa_+ g_a(\eta)]\, EP_1E^{-1} \right. \\ &\quad \times {}^{\bar{\mathrm{t}}}[(\zeta - \lambda_1(\eta))\, EM'(\eta; \zeta)]\, \tilde{f}_a(\xi, \bar{\zeta})\}_{\zeta=\lambda\pm i\kappa} + o_\kappa(1) \qquad \text{(B.18)} \\ &= \chi_{(\delta,N)}(c|\eta|) \int_{c|\eta|-\eta}^{c|\eta|+v} d\lambda\, \frac{\kappa}{(\lambda - c|\eta|)^2 + \kappa^2}\, {}^{\bar{\mathrm{t}}}[\Phi_3\, g_a(\eta)]\, EP_1E^{-1} \\ &\quad \times {}^{\bar{\mathrm{t}}}[(\lambda - c|\eta|)\, EM'(\eta; \lambda \pm i0)]\, \tilde{f}_a(\xi, \lambda \mp i0) + o_\kappa(1), \end{aligned}$$

where $(\lambda - c|\eta|)\, EM'(\eta; \lambda \pm i0)$ is a continuous function of λ on $[c|\eta| - \nu, c|\eta| + \nu]$, $\eta \in T(\sigma; \delta, N)$, if we define

$$\chi_{R_\mp}(\rho)\left[\frac{\lambda - c|\eta|}{\rho + \tau(\xi, \lambda \pm i0)}\right]_{\lambda = c|\eta|} \equiv \chi_{R_\mp}(\rho)\frac{1}{\partial_\lambda \tau(\xi, c|\eta| \pm i0)}$$
$$= \pm\chi_{R_\mp}(\rho)\, c\,|\rho|\,|\eta|^{-1},$$

$$\chi_{R_\mp}(\rho)\left[\frac{\lambda - c|\eta|}{\rho - \tau(\xi, \lambda \pm i0)}\right]_{\lambda = c|\eta|} = \mp\chi_{R_\pm}(\rho)\, c\,|\rho|\,|\eta|^{-1}.$$

We have (see (4.40))

$$[(\lambda - c|\eta|)EM'(\eta; \lambda \pm i0)_{\lambda = c|\eta|} = -[\chi_{R_\pm}(\rho)EP_1(\omega) + \chi_{R_\mp}(\rho)C'_\pm(\omega')EP_1(\omega)],$$
$$\chi_{R_\pm}(\rho)\tilde{f}_a(\xi, c|\eta| \mp i0) = \Phi_3\chi_+ f_a(\eta), \qquad \chi_{R_\mp}(\rho)\tilde{f}_a(\xi, c|\eta| \mp i0) = \Phi_3\chi_+ f_a(\tilde{\eta}),$$
$$\tilde{\eta} = (\xi, -\rho),$$

and hence from (B.17), (4.39)

$$h_2(\eta, \pm 0) = -\pi\chi_{(\delta,N)}(c|\eta|)\,{}^{t}[\Phi_3\chi_+ g_a(\eta)]\, EP_1[\chi_{R_\pm}(\rho)\, P_1(\omega)\, \Phi_3\chi_+ f_a(\eta)$$
$$+ \chi_{R_\mp}(\rho)\, P_1(\omega)\, C_\mp(\omega')\, \Phi_3\chi_+ f_a(\tilde{\eta})]. \tag{B.19}$$

Similarly,

$$h_3(\eta, \pm 0) = -\pi\chi_{(\delta,N)}(c|\eta|)\{{}^{\bar{t}}[\Phi_3\chi_+ g_a(\eta)]\, EP_1(\omega)\,\chi_{R_\pm}(\rho)$$
$$+ {}^{\bar{t}}[\Phi_3\chi_+ g_a(\eta)]\, EC_\pm(\omega')\, P_1(\omega)\,\chi_{R_\mp}(\rho)\}\, \Phi_3\chi_+ f_a(\eta), \tag{B.20}$$

and, finally,

$$h_4(\eta, \pm 0) = \pi\chi_{(\delta,N)}(c|\eta|)\{{}^{\bar{t}}[\Phi_3\chi_+ g_a(\eta)]\, EP_1(\omega)\, \Phi_3\chi_+ f_a(\eta)\,\chi_{R_\pm}(\rho)$$
$$+ {}^{\bar{t}}[\Phi_3\chi_+ g_a(\tilde{\eta})]\, C_\pm(\omega')\, EP_1(\omega)\, C_\mp(\omega')\, \Phi_3\chi_+ f_a(\tilde{\eta})\}. \tag{B.21}$$

Hence, from (B.17)–(B.21)

$$\pi^{-1}h(\eta, \pm 0) = \chi_{(\delta,N)}(c|\eta|)\,\chi_{R_\mp}(\rho)\{{}^{\bar{t}}[\Phi_3\chi_+ g_a(\eta)]\, EP_1(\omega)\, \Phi_3\chi_+ f_a(\eta)$$
$$- {}^{\bar{t}}[\Phi_3\chi_+ g_a(\eta)]\, EP_1(\omega)\, C_\mp(\omega')\, \Phi_3\chi_+ f_a(\tilde{\eta})$$
$$- {}^{\bar{t}}[\Phi_3\chi_+ g_a(\eta)]\, EC_\pm(\omega')\, P_1(\omega)\, \Phi_3\chi_+ f_a(\eta)$$
$$+ {}^{\bar{t}}[\Phi_3\chi_+ g_a(\tilde{\eta})]\, C_\pm(\omega')\, EP_1(\omega)\, C_\mp(\omega')\, \Phi_3\chi_+ f_a(\tilde{\eta})\}$$
$$= \chi_{(\delta,N)}(c|\eta|)\,{}^{\bar{t}}[\Psi_1 g(\eta)]\, E\Psi'_1 f(\eta)$$

by (4.39), (4.40), and (4.42). Hence, in $\lim_{\kappa\downarrow 0}$ the first term of (4.35) yields

$2i\pi$ times the second term of (4.53), and, similarly, the second term of (4.35) gives the third term of (4.53).

We now evaluate $\lim_{\kappa\downarrow 0}$ of $\Sigma_K(\kappa)$, $\Sigma_M(\kappa)$ of (B.14). We write

$$\Sigma_E(\kappa)=\int_{\{|\rho|\geqslant\sigma\}}\sigma_E(\eta,\kappa;\delta,N)\,d\eta+o_\sigma(1), \tag{B.22}$$

where

$$\sigma_E(\eta,\kappa;\delta,N)=\int_{c\delta}^{cN}d\lambda$$
$$\times\frac{{}^{\bar t}\{[\bar e_\pm(c^{-1}\lambda)-\bar e(\xi)]\,\Psi(e_\pm(c^{-1}\lambda))\,g(\eta)\}\,E[e_\mp(c^{-1}\lambda)-e(\xi)]\,\Psi'(\bar e_\mp(c^{-1}\lambda))\,f(\eta)}{[(\lambda-c\,|\eta|)^2+\kappa^2]\,\kappa^{-1}}.$$

The numerator here is a continuous function of $\lambda\in(c\delta,cN)$ for $\kappa=0$ if at the zero $\lambda=c\,|\xi|$ of $\Delta_E(\xi,e(c^{-1}\lambda))$ we define $\Delta_E(\xi,q\lambda)(q\lambda-cq\,|\xi|)^{-1}$ to be $\partial_\zeta\Delta_E(\xi,e(\xi))$ (cf. (B.15)). Evaluating now in (4.32)–(4.34), we have

$$\begin{aligned}
&\sigma_E(\eta,0^+;\delta,N)\\
&\quad=\pi\chi_{(\delta,N)}(|\xi|)\lim_{\lambda\to|\xi|}{}^{\bar t}[c\bar q(\lambda-|\xi|)\,\Psi(cq\lambda)\,g(\eta)]\,Ecq(\lambda-|\xi|)\,\Psi'(c\bar q\lambda)\,f(\eta)\\
&\quad=\pi\chi_{(\delta,N)}(|\xi|)[\rho+\tau(\xi,e(\xi))]^{-1}\,[\rho+\bar\tau(\xi,\bar e(\xi))]^{-1}\\
&\qquad\times{}^{\bar t}\tilde g(\xi,e(\xi))\,E(\{[cq\lambda-e(\xi)]\,C(\xi,\lambda)\,P(\xi,cq\lambda,-\tau(\xi,e(\xi))\}\,E^{-1}\\
&\qquad\times{}^{\bar t}\{[c\bar q\lambda-\bar e(\xi)]\,EC'(\xi,c\bar q\lambda)\,P(\xi,c\bar q\lambda,-\tau(\xi,\bar e(\xi)))_{\lambda=c|\xi|}\,\tilde f(\xi,\bar e(\xi))\\
&\quad\equiv\pi\chi_{(\delta,N)}(|\xi|)[\rho^2-\tau^2(\xi,e(\xi))]^{-1}\,F(\xi),
\end{aligned}$$

since $\bar\tau(\xi,\bar e(\xi))=-\tau(\xi,e(\xi))$ by (1.10). We set

$$\begin{aligned}
\sigma_E(\xi,0^+;\delta,N)&=\int_{-\infty}^{\infty}\sigma_E(\eta,0^+;\delta,N)\,d\rho\\
&=(2\pi i)^{-1}\,(2\tau_e)^{-1}\,\chi_{(\delta,N)}(|\xi|)\,F(\xi).
\end{aligned}\tag{B.24}$$

Now, performing the matrix multiplications indicated in $F(\xi)$ of (B.23), using (B.24), and letting $\sigma\downarrow 0$ in (B.22) we obtain (see (4.50), (4.52))

$$\Sigma_E(0^+)=\pi\int_{R^2}\chi_{(\delta,N)}(|\xi|)\,{}^{\bar t}[\Sigma_E\,g(\xi)]\,E\Sigma_E'\,f(\xi)\,d\xi$$

which on multiplying by $2i$ gives $2i\pi$ times the fourth term of (4.53). Similarly, $\Sigma_M(0^+)$ yields the fifth term of (4.53).

References

1. L. M. Brekhovskikh, "Waves in Layered Media," Izd. Akad. Nauk SSSR, Moscow, 1957.
2. V. A. Fock, "Electromagnetic Diffraction and Propagation Problems," Pergamon, London, 1965.
3. I. M. Gel'fand and G. E. Shilov, "Verallgemeinerte Funktionen," Bd. II, Deut. Verlag Wissenschaften, Berlin, 1962.
4. D. S. Gilliam and J. R. Schulenberger, A class of symmetric hyperbolic systems with special properties, *Comm. Partial Differential Equations* **4**, (1979), 509–536.
5. D. S. Gilliam and J. R. Schulenberger, On the structure of surface waves, *Adv. Appl. Math.*, to appear.
6. D. S. Gilliam and J. R. Schulenberger, Wave Propagation in Layered Media, II, "Electromagnetic Waves Over a Three-Dimensional Conducting Earth," Res. Rep. No. 4, AN AH Corporation, 1981.
7. J. A. Goldstein, Semigroups of operators and abstract Cauchy problems, Lecture Notes, Tulane University, New Orleans, La., 1970.
8. L. D. Landau and E. M. Lifschitz, "Electrodynamics of Continuous Media," Gos. Izd. Fiz.-Mat. Lit., Moscow, 1959.
9. J. L. Lions and E. Magenes, "Problèmes aux Limites non Homogènes," Vol. 1, Dunod, Paris, 1968.
10. R. S. Phillips, On dissipative operators, Lecture Series in Differential Equations, Vol. 2 (A. K. Aziz, Ed.), Von Nostrand–Reinhold, Princeton, N. J., 1969.
11. J. R. Schulenberger, On conservative boundary conditions for operators of constant deficit: the Maxwell operator, *J. Math. Anal. Appl.* **48** (1974), 223–248.
12. J. R. Schulenberger, Boundary waves on perfect conductors, *J. Math. Anal. Appl.* **66** (3) (1978), 514–549.
13. J. R. Schulenberger, Elastic waves in R^2_+, *J. Differential Equations* **29** (3) (1978), 405–438.
14. J. R. Schulenberger, "Wave Propagation in Layered Media, I, "Electromagnetic Waves Over a Dielectric-Clad Perfect Conductor in Three Dimensions," Res. Rep. No. 2, AN AH Corporation, (1980).